

Hotelling-Downs Model with Limited Attraction

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ABSTRACT

In this paper we study variations of the standard Hotelling-Downs model of spatial competition, where each agent attracts the clients in a restricted neighborhood, and each client randomly picks one attractive agent for service.

Two utility functions for agents are considered: support utility and winner utility. We generalize the results in [9] to the case where the clients are distributed arbitrarily. In the support utility setting, we show that a pure Nash equilibrium always exists by formulating the game as a potential game. In the winner utility setting, we show that there exists a Nash equilibrium in two cases: when there are at most 3 agents and when the size of attraction area is at least half of the entire space. We also consider the price of anarchy and the fairness of equilibria under certain conditions.

1. INTRODUCTION

Ever since the seminal works by Hotellings [12] and Downs [5], the Hotelling-Downs model has been applied to many problems, ranging from determining the standpoint of a election candidate to choosing locations for commercial facilities [14, 2, 21]. In the model, two firms choose shop locations in a street. Customers are distributed along the line. Assume the products of the firms are equal, so that the customers always go to the closer shop. Hence, one firm can always attract more customers by moving towards the competitor's location. Therefore, both firms choose the median point in the unique stable equilibrium, attracting an equal number of customers. This also sheds light upon the phenomenon that the candidates' standpoints in a political election are often close.

In political election scenarios, the assumption is taken to model the standpoints of the candidates in [20, 6] that every candidate has an ideal location in mind and cares about how close the winner's location is to the ideal location. Models with runoff voting are studied in [16, 3, 22], where the voting takes place in multiple rounds and only a subset of candidates from the previous round enter the next round. In commercial facility location problems, more complex models are proposed to address other issues. For example, when making decisions, customers consider the transportation cost caused by getting service. Such competition that involves both

location and pricing are considered by Hotelling's original model [12], and some other existing works [4, 19, 23, 15, 8].

In this paper, we focus on the pure location game [1]. It sacrifices non-existence of Nash equilibria in the original pure location game. Eaton *et al.* [7] first show there is no Nash equilibrium when there are 3 agents in the one-dimensional space. Shaked [17] extends the non-existence to two-dimensional space. Thereafter Osborne *et al.* [13] show the Nash equilibrium does not exist in a wide range of settings when there are more than 2 agents. However, a mixed Nash equilibrium is guaranteed to exist [4, 18]. This result is not obvious considering that the utility functions in these games are not continuous with the action. Generally, a mixed Nash equilibrium is not guaranteed to exist in such games.

The original Hotelling-Downs model suffers from some problematic assumptions: customers always choose the nearest shop without considering the distance, contradicting to the fact that a shop is no more attractive to a customer if it is too far away. Furthermore, customers choose the shop without considering competing shops, while in daily life, it is hard to say which shop attracts more customers if two shops are close enough with similar products. These issues are also discussed in [9].

To address the above issues, we consider the Hotelling-Downs model with limited attraction, proposed in [9]. In this model, all firms (called agents hereafter) only attract customers (called clients hereafter) in a limited distance, and if a client is attracted by multiple agents, the client picks one from those agents with equal chance.

We analyze the Nash equilibria with two utility functions for agents: support utility and winner utility (i.e., winner takes all setting in [9]). In the support utility setting, agents focus on maximizing the number of its clients, modeling the commercial competitions. While in the winner utility setting, the winner in the competition takes all the utility, which is typical for the political voting.

We extend the results on uniform distribution in [9] to arbitrary distributions. First of all, we consider the existence of *pure* Nash equilibria. In the support utility setting, when the distribution is uniform and agents have the same attracting distance (called width hereafter), the existence of Nash equilibria can be shown by simply constructing one. However, this method does not work any more under other distribution. We solve this problem by formulating it as a potential game. In winner utility maximization setting, to our knowledge, there is no standard tool to guarantee the

existence of Nash equilibria. However, we show that a pure Nash equilibrium does exist in some simple cases.

Secondly, we study fairness[10] and the price of anarchy of Nash equilibria in the support utility setting (both of which are straightforward in the winner utility setting). Fairness characterizes how fairly the utilities are divided among all agents. The price of anarchy measures how efficiency decreases due to agents' selfish behaviors. We give tight bounds on both criteria.

1.1 Results and Contributions

- In the support utility maximization setting, support utility is continuous. Applying Glicksberg's theorem [11], this continuous game guarantees a mixed Nash equilibrium. If we let the agents dynamically best respond to the other's locations (one agent each round), then the location profile converges to a Nash equilibrium. Scrutinizing each agent's action, each improvement actually increases a potential function. Thus the game admits a pure Nash equilibrium.
- In the winner utility maximization setting, the winner utility is not continuous any more. We restrict the problem to the case when the agents have the same width. We prove that when there are at most 3 agents, there exists a pure Nash equilibrium. The three-agents case is very special in other variants [7, 17, 18]. The existence of Nash equilibria complements their results. Moreover, if the agent's width is at least half of the client space, then there also exists a Nash equilibrium for any number of agents.
- We study fairness and the price of anarchy of Nash equilibria in the support utility setting (both of which are straightforward in the winner utility setting). We show that the fairness criterion can be bounded by $\frac{1}{2\lceil w_M/w_m \rceil}$, where $w_M = \max\{w\}$ and $w_m = \min\{w\}$. We also prove that the price of anarchy at least $\frac{1}{2}$. Both bounds are tight.

The structure of this paper is as follows: In Section 2, we describe the coined Hotelling-Downs model with limited attraction in [9]. In Section 3, we prove the existence of the Nash equilibria in the support-utility maximization setting. In Section 4, we construct a Nash equilibrium in winner utility maximization setting. In Section 5, the support utilities in Nash equilibria are compared. In Section 6, the price of anarchy is given and we give an upper bound on the amount of clients that have not been served.

2. HOTELLING-DOWNS MODEL WITH LIMITED ATTRACTION

We consider a one-dimensional location space, represented by the interval $[0, 1]$. A continuum of clients are distributed in the interval according to some density function $f(x)$. Let $N = (1, 2, \dots, n)$ denote a finite set of agents and each agent i is associated with an attraction width w_i . Each agent chooses a location in $[0, 1]$ and an attraction interval R_i centered at the chosen location is formed. The agent obtains the support from the clients in his attraction interval. If a client is covered by multiple agents, the client simply randomly choose one, i.e. the support of this client is equally

divided among these agents in expectation. Assume that agent i chooses location x_i then the attraction interval R_i is $[x_i - w_i/2, x_i + w_i/2]$. We assume that $f(x) = 0$ outside the interval $[0, 1]$ and thus each agent will only choose a location from $[\frac{w_i}{2}, 1 - \frac{w_i}{2}]$.

Let \vec{x} denote the joint location profile (x_1, x_2, \dots, x_n) , and \vec{x}_{-i} denote the profile without i , i.e. $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. Given \vec{x} , let congestion function $c(x, \vec{x})$ be the the number of attraction intervals covering point x ,

$$c(x, \vec{x}) = \#\{x_i | x \in R_i\}$$

Clearly, the following equation holds:

$$c(x, \vec{x}) = \begin{cases} c(x, \vec{x}_{-i}) & x \notin [x_i - \frac{w_i}{2}, x_i + \frac{w_i}{2}] \\ c(x, \vec{x}_{-i}) + 1 & x \in [x_i - \frac{w_i}{2}, x_i + \frac{w_i}{2}] \end{cases}$$

For simplicity, we use $c(x)$ instead of $c(x, \vec{x})$ when there is no ambiguity.

If a point x is covered by multiple attraction intervals (i.e., $c(x) \geq 2$), then the support of that point is evenly divided among all these agents. Agent i 's support s_i is then defined to be the total support from his attraction interval:

$$s_i(\vec{x}) = \int_{x_i - w_i/2}^{x_i + w_i/2} \frac{f(x)}{c(x)} dx$$

In our model, we assume that the distribution function $f(x)$ and the width $w_i, \forall i$ are publicly known. We consider two kinds of utility settings: support utility and winner utility. The support utility setting uses the support function as agents' utility function. In the winner utility setting, only agents with the largest support are considered to be the winners, and share a total utility of 1 equally among them, while other agents have utility 0. Note that in the winner utility setting, each agent only cares about whether he is a winner and the number of winners, since if the agent is a winner, he has a higher utility when there are less winners.

Formally, an attraction game is defined as follows:

DEFINITION 1. Given the clients' distribution $f(x)$, an attraction game is a tuple $G = (N, w, L, u)$, where:

- $N = (1, 2, \dots, n)$ is the set of all agents;
- $w = (w_1, w_2, \dots, w_n)$ is the widths associated to agents;
- $L = L_1 \times L_2 \times \dots \times L_n$ is the set of all possible location profiles, where $L_i = [\frac{w_i}{2}, 1 - \frac{w_i}{2}]$.
- $u = (u_1, u_2, \dots, u_n)$ is the utility functions for the agents, the definition of which depends on the setting we consider:

- in the support utility setting, $u_i(\vec{x}) = s_i(\vec{x})$;
- in the winner utility setting,

$$u_i(\vec{x}) = \begin{cases} \frac{1}{|W|} & i \in W \\ 0 & \text{otherwise} \end{cases}$$

where W denotes the set of winners.

A Nash equilibrium of game G is a stable location profile, where no agent can deviate to another location to get a higher utility.

DEFINITION 2 (NASH EQUILIBRIUM). Given a game G , the set of Nash equilibria $NE(G)$ contains all location profiles \vec{x} , such that $\forall i \in N$ and $\forall x'_i \in L_i$,

$$u_i(x_i, \vec{x}_{-i}) \geq u_i(x'_i, \vec{x}_{-i})$$

In different utility settings, the definitions of the utility functions are different, and thus have distinct Nash equilibria. Consider the following example:

Example. Assume that the clients are distributed uniformly. There are 3 agents with widths $w_1 = 0.4$, $w_2 = 0.3$, $w_3 = 0.4$. The location profile is $\vec{x} = (0.2, 0.65, 0.8)$ (see Figure 1).

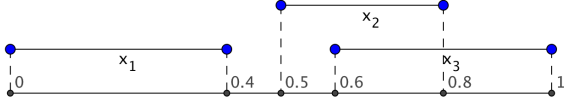


Figure 1: example

The corresponding congestion function is

$$c(x) = \begin{cases} 1 & x \in [0, 0.4] \\ 0 & x \in (0.4, 0.5) \\ 1 & x \in [0.5, 0.6] \\ 2 & x \in [0.6, 0.8] \\ 1 & x \in (0.8, 1] \end{cases}$$

The support of the 3 agents are

- $s_1 = \int_0^{0.4} f(x) dx = 0.4$
- $s_2 = \int_{0.5}^{0.6} f(x) dx + \int_{0.6}^{0.8} \frac{f(x)}{2} dx = 0.2$
- $s_3 = \int_{0.6}^{0.8} \frac{f(x)}{2} dx + \int_{0.8}^1 f(x) dx = 0.3$

In the support utility setting, the profile \vec{x} does not form a Nash equilibrium, since given \vec{x}_{-2} , agent 2 has incentive to deviate to 0.55. And by doing so, u_2 increases as R_2 has less intersection with R_3 . However, in the winner utility setting, the profile \vec{x} forms a Nash equilibrium and agent 1 is the unique winner.

3. EXISTENCE OF NASH EQUILIBRIUM IN THE SUPPORT UTILITY SETTING

A well known theorem of Glicksberg [11] states that every continuous game has a mixed Nash equilibrium. By definition,

$$u_i(x_i, \vec{x}_{-i}) = \int_{x_i - \frac{w_i}{2}}^{x_i + \frac{w_i}{2}} \frac{f(x)}{c(x, \vec{x}_{-i}) + 1} dx$$

Agent i 's support utility is continuous with x_i . According to Glicksberg's theorem, there exists a mixed Nash equilibrium in this setting.

However, due to the special structure of our model, we could further show that a pure Nash equilibrium always exists. Under the support utility setting, the game can be viewed as a congestion game where the resources are the densities associated to each point. It is known that every finite congestion game has a pure strategy Nash equilibrium.

Although agents's action space is infinitely in the game, we can still show that a pure strategy Nash equilibrium exists.

THEOREM 1. There exists a pure Nash equilibrium in the support utility setting.

PROOF. Given other agents' locations \vec{x}_{-i} , agent i 's support utility can be written as

$$u_i(x_i, \vec{x}_{-i}) = \int_{x_i - \frac{w_i}{2}}^{x_i + \frac{w_i}{2}} \frac{f(x)}{c(x, \vec{x}_{-i}) + 1} dx$$

If agent i prefers x'_i to x_i , we have

$$\int_{x'_i - \frac{w_i}{2}}^{x'_i + \frac{w_i}{2}} \frac{f(x)}{c(x, \vec{x}_{-i}) + 1} dx > \int_{x_i - \frac{w_i}{2}}^{x_i + \frac{w_i}{2}} \frac{f(x)}{c(x, \vec{x}_{-i}) + 1} dx$$

On both sides of the inequality, we add the following term

$$\int_0^1 \sum_{k=1}^{c(x, \vec{x}_{-i})} \frac{f(x)}{k} dx$$

The left side of the above inequality becomes

$$\begin{aligned} & \int_0^1 \sum_{k=1}^{c(x, \vec{x}_{-i})} \frac{f(x)}{k} dx + \int_{x'_i - \frac{w_i}{2}}^{x'_i + \frac{w_i}{2}} \frac{f(x)}{c(x, \vec{x}_{-i}) + 1} dx \\ &= \int_0^{x'_i - \frac{w_i}{2}} \sum_{k=1}^{c(x, \vec{x}_{-i})} \frac{f(x)}{k} dx + \int_{x'_i + \frac{w_i}{2}}^1 \sum_{k=1}^{c(x, \vec{x}_{-i})} \frac{f(x)}{k} dx \\ & \quad + \int_{x'_i - \frac{w_i}{2}}^{x'_i + \frac{w_i}{2}} \sum_{k=1}^{c(x, (x'_i, \vec{x}_{-i}))} \frac{f(x)}{k} dx \\ &= \int_0^1 \sum_{k=1}^{c(x, (x'_i, \vec{x}_{-i}))} \frac{f(x)}{k} dx \end{aligned}$$

The right side is similar and the inequality becomes

$$\int_0^1 \sum_{k=1}^{c(x, (x_i, \vec{x}_{-i}))} \frac{f(x)}{k} dx > \int_0^1 \sum_{k=1}^{c(x, \vec{x})} \frac{f(x)}{k} dx \quad (1)$$

If we start from an arbitrary location profile, many agents' strategies are not optimal. Then in each round, one agent chooses a better position according to other agents' strategies. By equation (1), each time one agent improves his support utility, he actually improves a potential function,

$$\Phi(\vec{x}) = \int_0^1 \sum_{k=1}^{c(x, \vec{x})} \frac{f(x)}{k} dx$$

The potential function can be upper bounded,

$$\Phi(\vec{x}) = \int_0^1 \sum_{k=1}^{c(x, \vec{x})} \frac{f(x)}{k} dx \leq \int_0^1 \sum_{k=1}^n \frac{f(x)}{k} dx = \sum_{k=1}^n \frac{1}{k}.$$

Combined the fact $\Phi(\vec{x})$ is continuous with \vec{x} , we have that $\Phi(\vec{x})$ has a maximum value. In the location profile $\vec{x}^* = \arg \max_{\vec{x}} \Phi(\vec{x})$, no agent can improve his support utility and thus it is a Nash equilibrium. \square

Noticing that the above proof does not use the fact of one-dimension space, and the result can be extended to multi-dimensional space.

COROLLARY 1. There is a pure Nash equilibrium in the support utility maximization setting if the location space is multi-dimensional.

4. EXISTENCE OF NASH EQUILIBRIUM IN THE WINNER UTILITY SETTING

In the winner utility setting, the utility of agent i is no longer continuous with respect to the agent's location. The potential function in the support-utility-maximizing setting does not work.

DEFINITION 3. A game is an ordinal potential game, if there is a function $\phi : A \rightarrow R$ such that $\forall a_{-i} \in A_{-i}, \forall a'_i, a''_i \in A_i,$

$$u_i(a'_i, a_{-i}) - u_i(a''_i, a_{-i}) > 0 \Leftrightarrow \Phi(a'_i, a_{-i}) - \Phi(a''_i, a_{-i}) > 0$$

It seems difficult to design a potential function to prove NE existence, the reason is the following theorem.

THEOREM 2. Winner utility maximization game is not an ordinal potential game.

We prove by contradiction.

PROOF. Consider the distribution function is

$$f(x) = \begin{cases} 4/3 & x \in [0, 1/3) \\ 1/3 & x \in [1/3, 2/3) \\ 4/3 & x \in [2/3, 1] \end{cases}$$

There are three agents with same width $w = 1/3$. We give two different paths from the location profile $(1/6, 1/6, 1/6)$ to $(1/6, 1/6, 5/6)$.

Path 1: $(1/6, 1/6, 1/6) \rightarrow (1/6, 1/6, 5/9) \rightarrow (1/6, 5/9, 5/9) \rightarrow (1/6, 5/9, 5/6) \rightarrow (1/6, 1/6, 5/6)$

Path 2: $(1/6, 1/6, 1/6) \rightarrow (1/6, 1/6, 5/6)$.

Suppose there exists a potential function Φ . In Path 1, u_3 decreases in the first step, u_2 decreases in the second step. The support utility of the deviating agent does not change in the following steps. By definition of Φ , we should have $\Phi(1/6, 1/6, 5/6) < \Phi(1/6, 1/6, 1/6)$.

In Path 2, u_3 increases in the first step. By definition, we should have $\Phi(1/6, 1/6, 5/6) > \Phi(1/6, 1/6, 1/6)$, contradiction. \square

In this setting, there exists a new strategy that an agent increase winner utility by decreasing the support utility of both winner and himself. Consider the following example.

Example. Let the distribution $f(x)$ be

$$f(x) = \begin{cases} 5/4 & x \in [0, 0.4] \\ 5/6 & x \in (0.4, 1] \end{cases}$$



Figure 2: example

There are two agents, and the location profile is $\vec{x} = (0.2, 0.8)$. The width are equal $w_1 = w_2 = 0.4$. In this case, agent 1 is the winner and $u_1 = 0.5, u_2 = 10/3$. However, agent 2 can move to x_1 and share the support from $[0, 0.4]$ with agent 1. These two agents' support utility become 0.25 and both agents are winners. Notice that agent 2 becomes a winner by decreasing both agents' support utility.

When we consider winner utility maximization setting, we restrict to the case all agents have the same width $w_i = w$. First we prove a lemma which will be used frequently. This lemma roughly gives a situation where two agents have no incentive to deviate.

LEMMA 1. Fix $k \geq 0$ agents' locations \vec{x} at first¹. Let X be the set of maximizers of

$$\int_{x-\frac{w}{2}}^{x+\frac{w}{2}} \frac{f(y)}{c(y, \vec{x}) + 1} dy$$

Suppose there are two new agents A and B . If both two agents choose the two locations $x_A, x_B \in X$ (x_A and x_B could be the same) simultaneously, then both agents have the same support utility, and both A and B cannot have more support utility than the other by changing location.

PROOF. By definition of x_A and x_B , we have

$$\int_{x_A-\frac{w}{2}}^{x_A+\frac{w}{2}} \frac{f(x)}{c(x) + 1} dx = \int_{x_B-\frac{w}{2}}^{x_B+\frac{w}{2}} \frac{f(x)}{c(x) + 1} dx.$$

When A and B are located at the same time, their attraction interval may overlap. This will decrease the support from clients in the intersection interval, but the decrements in two support utility are the same. We use R_A denote the attraction interval $[x_A - \frac{w}{2}, x_A + \frac{w}{2}]$ and R_B denote the interval $[x_B - \frac{w}{2}, x_B + \frac{w}{2}]$. Formally, agent A 's support utility will be

$$\begin{aligned} & \int_{R_A - R_B} \frac{f(x)}{c(x) + 1} dx + \int_{R_A \cap R_B} \frac{f(x)}{c(x) + 2} dx \\ &= \int_{R_A} \frac{f(x)}{c(x) + 1} dx - \int_{R_A \cap R_B} f(x) \left(\frac{1}{c(x) + 2} - \frac{1}{c(x) + 1} \right) dx \\ &= \int_{R_B} \frac{f(x)}{c(x) + 1} dx - \int_{R_A \cap R_B} f(x) \left(\frac{1}{c(x) + 2} - \frac{1}{c(x) + 1} \right) dx \\ &= \int_{R_B - R_A} \frac{f(x)}{c(x) + 1} dx + \int_{R_A \cap R_B} \frac{f(x)}{c(x) + 2} dx. \end{aligned}$$

which is same as agent B 's support utility.

For the second part, we prove by contradiction. Suppose A moves to $x_{A'}$ and gets more support utility than B , then

$$\begin{aligned} & \int_{R_{A'} - R_B} \frac{f(x)}{c(x) + 1} dx + \int_{R_{A'} \cap R_B} \frac{f(x)}{c(x) + 2} dx > \\ & \int_{R_B - R_{A'}} \frac{f(x)}{c(x) + 1} dx + \int_{R_{A'} \cap R_B} \frac{f(x)}{c(x) + 2} dx \\ \Rightarrow & \int_{R_{A'} - R_B} \frac{f(x)}{c(x) + 1} dx + \int_{R_{A'} \cap R_B} \frac{f(x)}{c(x) + 1} dx > \\ & \int_{R_B - R_{A'}} \frac{f(x)}{c(x) + 1} dx + \int_{R_{A'} \cap R_B} \frac{f(x)}{c(x) + 1} dx \\ \Rightarrow & \int_{R_{A'}} \frac{f(x)}{c(x) + 1} dx > \int_{R_B} \frac{f(x)}{c(x) + 1} dx \end{aligned}$$

which contradicts to the fact that x_B is a best location. Since A and B are symmetric, agent B cannot get more support utility than A neither. \square

When there are 2 agents, we put both agents at the same position where maximizes $\int_{[x-w/2, x+w/2]} f(y) dy$. By setting $k = 0$ in Lemma 1, we know both agents are winners. Either one cannot become the unique winner by deviating. Thus the location profile constitutes a Nash equilibrium.

¹This does not need to be a Nash equilibrium.

THEOREM 3. *There is a pure Nash equilibrium when there are 2 agents.*

When there are 3 agents, the problem becomes quite complicated. Since there is no symmetric property, there are many cases to consider when we are checking the stable equilibrium. We propose Algorithm 1 for 3 agents.

- Let u_1 denote the largest support utility that the first player could achieve, i.e. u_1 is the maximum value of $\int_{x-w/2}^{x+w/2} f(y)dy$.
- *Case 1:* If we can allocate three agents at the same time such that everyone achieves the support utility u_1 , then we allocate them at those three locations.
- *Case 2:* If we can allocate only two agents at the same time such that every one achieves the support utility u_1 , then we allocate agent 1 at one of the two locations, the other two agents together at the other location.
- Otherwise, we can allocate only one agent that achieves the support utility u_1 . There exists a set of locations that agent 1 achieves the support utility u_1 , we allocate agent 1 at the leftmost one, denoted by x_1 . Given agent 1's location, let the largest support utility for agent 2 be u_2 .
 - *Case 3:* If agent 2 can achieve support utility u_2 at location x_1 , we allocate agents 2 and 3 together at x_1 .
 - *Case 4:* Otherwise, if on both left side and right side of agent 1, there exists locations where agent 2 achieves support utility u_2 . We allocate agent 2 at the rightmost position in the left part and agent 3 at the leftmost position in the right part, i.e.,

$$x_2 = \max_{x < x_1} \left\{ x \left| \int_{x-w/2}^{x+w/2} \frac{f(y)}{c(y)+1} dy = u_2 \right. \right\},$$

$$x_3 = \min_{x > x_1} \left\{ x \left| \int_{x-w/2}^{x+w/2} \frac{f(y)}{c(y)+1} dy = u_2 \right. \right\}.$$

- *Case 5:* Otherwise, the locations that maximize agent 2's support utility lie on the same side of agent 1. We allocate agent 2 to the closest position and agent 3 at the farthest position. Let

$$t_2 = \min \left\{ x \left| \int_{x-w/2}^{x+w/2} \frac{f(y)}{c(y)+1} dy = u_2 \right. \right\},$$

$$t_3 = \max \left\{ x \left| \int_{x-w/2}^{x+w/2} \frac{f(y)}{c(y)+1} dy = u_2 \right. \right\}.$$

If $t_2 < x_1$, then set $x_2 = t_3$, $x_3 = t_2$. Otherwise, set $x_2 = t_2$, $x_3 = t_3$.

THEOREM 4. *When there are 3 agents, Algorithm 1 gives a pure Nash equilibrium.*

Here is the intuition of Algorithm 1. Most of the time we allocate agent 1 where he gets the largest support utility. Then we allocate agent 2 and 3 to get the largest support

utility as possible. Agent 2 and 3 have the same support utility and hinder each other. If agent 2 wants to get the same support utility as agent 1 by decreasing both u_1 and u_2 , then u_3 becomes the largest, and agent 3 is the unique winner. Thus the location profile forms a Nash equilibrium.

PROOF. The proof follows algorithm's structure. In each case, we consider who wins and whether the agents' attraction intervals intersect. In all cases, we prove no one has incentive to deviate.

In Case 1, the winner set is $\{1, 2, 3\}$. Keeping agent 2's location fixed, by Lemma 1, agent 1 cannot get more support utility than agent 3.

Thus if agent 1 moves, the other two agents have at least the same support utility. Using similar arguments, no one gets more support utility than any other player. So on one has incentive to deviate.

In Case 2, winner set is $\{1\}$. Since agent 2 and 3 are at the same location, we only need to prove agent 2 does not have incentive to deviate. We prove by contradiction. Let agent 2 could become a winner by deviating to x'_2 . Let R'_2 denote the corresponding attraction interval. If R'_2 does not intersect with R_1 , agent 1 always has more support utility than agent 2. Then R'_2 intersects with R_1 . For same reason, R'_2 intersects with R_3 . Without agent 3, agent 2 has at most the same support utility as agent 1. But R_3 only has intersection with R'_2 , this intersection makes agent 2 have strictly less support utility than the agent 1. So agent 2 cannot become a winner and nobody has incentive to deviate.

In Case 3, winner set is $\{1, 2, 3\}$. Since they have the same location, we only need to prove agent 3 has no incentive to deviate. Suppose agent 3 has incentive to deviate to x'_3 with attraction interval R'_3 , then he must become the unique winner. Formally,

$$\begin{aligned} & \int_{R'_3 \cap R_1} \frac{f(x)}{3} dx + \int_{R'_3 - R_1} f(x) dx \geq \\ & \int_{R'_3 \cap R_1} \frac{f(x)}{3} dx + \int_{R_1 - R_3} \frac{f(x)}{2} dx \\ \Rightarrow & \int_{R'_3 \cap R_1} \frac{f(x)}{2} dx + \int_{R'_3 - R_1} f(x) dx \geq \\ & \int_{R'_3 \cap R_1} \frac{f(x)}{2} dx + \int_{R_1 - R_3} \frac{f(x)}{2} dx \end{aligned}$$

Consider the situation when only agent 1 has been located. The left side of the inequality is agent 2's support utility by choosing x'_3 . The right side of the inequality is agent 2's support utility by choosing x_1 . That means agent 2 gets more support utility by choosing x'_3 than choosing x_1 , contradicting to the assumption. Hence agent 3 has no incentive to deviate.

In Case 4, we have $u_2 = u_3$ by Lemma 1. There are 3 possibilities about who the winners are.

Case 4.1: Winner set is $\{2, 3\}$. Agent 2 and 3 have no incentive to deviate by Lemma 1. We next prove agent 1 has no incentive to deviate. If $R_2 \cap R_1 = \emptyset$, agent 3 cannot have more support utility than agent 1. So we have $R_2 \cap R_1 \neq \emptyset$ and $R_1 \cap R_3 \neq \emptyset$. We consider whether $R_2 \cap R_3$ is empty.

Case 4.1.1: $R_2 \cap R_3 \neq \emptyset$. Suppose agent 1 benefits by deviating to x'_1 . Let R'_1 denote the new attraction interval.

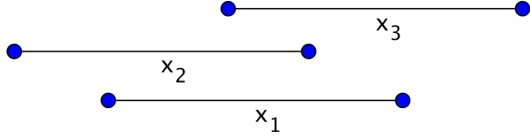


Figure 3: example

If $R'_1 \cap R_3 = \emptyset$, then we have

$$\begin{aligned}
u'_1 &= \int_{R'_1 - R_2} f(x) dx + \int_{R'_1 \cap R_2} \frac{f(x)}{2} dx \\
&\leq \int_{R'_1 - R_1} f(x) dx + \int_{R'_1 \cap R_1} \frac{f(x)}{2} dx \\
&\leq \int_{R_3 \cap R_1} \frac{f(x)}{2} dx + \int_{R_3 - R_1} f(x) dx \\
&< \int_{R_3 \cap R_2} \frac{f(x)}{2} dx + \int_{R_3 - R_2} f(x) dx
\end{aligned}$$

This is the agent 3's support utility after agent 1's deviation. Agent 1 cannot become a winner.

If $R'_1 \cap R_3 \neq \emptyset$ and $R'_1 \cap (R_3 - R_2) = \emptyset$, then we have

$$\begin{aligned}
u'_1 &= \int_{R'_1 - R_2} f(x) dx + \int_{R_2 - R_3} \frac{f(x)}{2} dx + \int_{R'_1 \cap R_3} \frac{f(x)}{3} dx \\
&\leq \int_{R_2 - R_3} \frac{f(x)}{2} dx + \int_{R'_1 - R_3} \frac{f(x)}{3} dx + \int_{R_2 - R'_1} \frac{f(x)}{2} dx \\
&< \int_{R_2 - R_1} f(x) dx + \int_{R_1 - R_3} \frac{f(x)}{2} dx + \int_{R'_1 - R_3} \frac{f(x)}{3} dx \\
&\quad + \int_{R_2 - R'_1} \frac{f(x)}{2} dx \\
&= \int_{R_3 - R_1} f(x) dx + \int_{R_1 - R_2} \frac{f(x)}{2} dx + \int_{R'_1 - R_3} \frac{f(x)}{3} dx \\
&\quad + \int_{R_2 - R'_1} \frac{f(x)}{2} dx
\end{aligned}$$

This is the agent 3's support utility after agent 1's deviation.

If $R'_1 \cap (R_3 - R_2) \neq \emptyset$. Since agent 1 has at least the same support utility as agent 2, we have

$$\int_{R'_1 - R_2} \frac{f(x)}{2} dx \geq \int_{R_2 - R'_1} f(x) dx,$$

contradicting to the definition of agent 2's location. To sum up, agent 1 cannot become a winner by deviating in Case 4.1.1.

Case 4.1.2: $R_2 \cap R_3 = \emptyset$. The proof is similar to the that in Case 4.1.1 and thus omitted.

Case 4.2: Winner set is $\{1\}$. Then agent 1 has no incentive to move. By definition of agent 2's location, we have

$$\begin{aligned}
\int_{R_2 - R_1} f(x) dx + \int_{R_1 \cap R_2} \frac{f(x)}{2} dx &> \int_{R_1} \frac{f(x)}{2} dx \\
\int_{R_2 - R_1} f(x) dx &> \int_{R_1 - R_2} \frac{f(x)}{2} dx
\end{aligned}$$

We claim that $R_1 \cap R_2 \cap R_3 = \emptyset$, otherwise we have

$$\begin{aligned}
u_2 &= \int_{R_2 - R_1} f(x) dx + \int_{R_2 - (R_1 \cap R_3)} \frac{f(x)}{2} dx \\
&\quad + \int_{R_2 \cap R_1 \cap R_3} \frac{f(x)}{3} dx \\
&> \int_{R_1 - R_2} \frac{f(x)}{2} dx + \int_{R_2 - (R_1 \cap R_3)} \frac{f(x)}{2} dx \\
&\quad + \int_{R_2 \cap R_1 \cap R_3} \frac{f(x)}{3} dx \\
&= u_1
\end{aligned}$$

Suppose agent 2 deviates to x'_2 . If $x'_2 < x_2$, u_2 weakly decreases and $R_1 \cap R_2$ weakly shrinks. Furthermore, u_1 weakly increases. Agent 2 would not be the winner. If $x'_2 \in (x_2, x_3)$, by the definition of x_2 and x_3 , agent 2 has less support utility than agent 3 no matter whether $R_2 \cap R_3$ is empty. If $x'_2 \in [x_3, 1)$, $R'_2 \cap R_3$. By Lemma 1, u'_2 is at most equal to agent 3's support utility. Now we prove agent 1 has strictly higher support utility than agent 3 after agent 2's deviation. Agent 1's support utility is

$$\begin{aligned}
&\int_{R_1 - R_3} f(x) dx + \int_{R_3 - R'_2} f(x) dx + \int_{R_1 \cap R_3 \cap R'_2} \frac{f(x)}{3} dx \\
&> \int_{R_3 - R_1} f(x) dx + \int_{R_3 - R'_2} \frac{f(x)}{2} dx + \int_{R_1 \cap R_3 \cap R'_2} \frac{f(x)}{3} dx \\
&> \int_{R_3 - R_1} \frac{f(x)}{2} dx + \int_{R_3 - R'_2} \frac{f(x)}{2} dx + \int_{R_1 \cap R_3 \cap R'_2} \frac{f(x)}{3} dx
\end{aligned}$$

This is the agent 3's support utility. To sum up, agent 2 would not deviate, neither does agent 3.

Case 4.3: Winner set is $\{1, 2, 3\}$. The proof of agent 1 would not deviate is similar to that in Case 4.1. The proof of agent 2 or 3 would not deviate is similar to that in Case 4.2.

Case 5: First we claim winner set can not be $\{2, 3\}$. By definition, we have $R_3 \cap R_1 \subset R_3 \cap R_2$. We consider u_1, u_2 when there are only agent 1 and 2, then we take count into the impact of agent 3.

$$\begin{aligned}
u_1 &= \int_{R_1 - R_2} f(x) dx + \int_{R_1 \cap R_2} \frac{f(x)}{2} dx \\
&\quad + \int_{R_1 \cap R_2 \cap R_3} \left(\frac{f(x)}{3} - \frac{f(x)}{2} \right) dx \quad (2) \\
u_2 &= \int_{R_2 - R_1} f(x) dx + \int_{R_1 \cap R_2} \frac{f(x)}{2} dx \\
&\quad + \int_{R_1 \cap R_2 \cap R_3} \left(\frac{f(x)}{3} - \frac{f(x)}{2} \right) dx \\
&\quad + \int_{(R_2 \cap R_3) - R_1} \left(\frac{f(x)}{2} - f(x) \right) dx \quad (3)
\end{aligned}$$

We have $u_1 \geq u_2$. Thus if agent 2 is a winner, agent 1 is a winner too.

Case 5.1: Winner set is $\{1\}$. In this sub-case, the argument is independent with the leftmost property of x_1 . Then w.l.o.g. we assume $x_1 < x_2 \leq x_3$. Agent 1 has no incentive to deviate. Suppose agent 2 deviate to x'_2 . If $x'_2 \in [0, x_2) \cup (x_3, 1]$, agent 3 has strictly more support utility than agent 2. If $x'_2 \in (x_2, x_3]$, agent 1's support utility weakly increase, agent 2's support utility is weakly smaller than agent 3's. Agent 3's support utility weakly decreases.

Then agent 1 has strictly more support utility than agent 2. Thus agent 2 would not deviate. Suppose agent 3 deviates to x'_3 . If $x'_3 \in [0, x_2) \cup (x_3, 1]$, then agent 2 has strictly more support utility than agent 3. If $x'_3 \in [x_2, x_3)$, agent 3 has weakly less support utility than agent 2. But the support utility difference between agent 2 and agent 1 becomes larger. Thus agent 3 would not deviate.

Case 5.2: Winner set is $\{1, 2, 3\}$. The proof that agent 2 and 3 have no incentive to deviate is same as in Case 5.1. Since $u_1 = u_2$ and by Equation (2) and (3), we have $R_2 \cap R_3 = \emptyset$. Since $u_1 = u_2$, x_2 is also a best choice for agent 1 at the beginning. By the leftmost property of x_1 , we know $x_1 < x_2$. Suppose agent 1 deviates to x'_1 . If $x'_1 \in [0, x_1) \cup (x_1, x_2)$, agent 1 has strictly less support utility than agent 2. If $x'_1 \in [x_2, 1)$ and $R'_1 \cap R_3 = \emptyset$, agent 1 has strictly less support utility than agent 3.

$$\begin{aligned} u'_1 &= \int_{R'_1 - R_2} f(x)dx + \int_{R'_1 \cap R_2} \frac{f(x)}{2} dx \\ &< \int_{R'_1 - R_1} f(x)dx + \int_{R'_1 \cap R_1} \frac{f(x)}{2} dx \leq u_3 \end{aligned}$$

If $x'_1 \in [x_2, 1)$ and $R'_1 \cap R_3 \neq \emptyset$,

$$\begin{aligned} u'_1 &= \int_{R'_1 - R_2 - R_3} f(x)dx + \int_{R'_1 \cap (R_2 \cup R_3)} \frac{f(x)}{2} dx \\ &< \int_{R'_1 - R_2} f(x)dx + \int_{R'_1 \cap R_2} \frac{f(x)}{2} dx \end{aligned}$$

This is agent 2's support utility after deviation. Agent 1 has strictly less support utility than agent 2. Hence in Case 5.2, no agent would deviate. \square

When $w = 0.5$, the attraction intervals overlap in general. We make use of this property and give Algorithm 2 to construct a Nash equilibrium with width 0.5.

- Let u_1 denote the largest support utility that agent 1 could achieve, i.e. $u_1 = \max\{\int_{[x-w/2, x+w/2]} f(y)dy\}$.
- If we can allocate 2 agents at the same time such that everyone achieves the support utility u_1 , i.e.

$$\int_{[0, 0.5]} f(x)dx = \int_{[0, 0.5]} f(x)dx = u_1.$$

When there are k agents, we allocate $\lceil k/2 \rceil$ agents at 0.25 and $k - \lceil k/2 \rceil$ agents at 0.75.

- If we can allocate only one agent that achieves the support utility u_1 , then we allocate the first agent at x_1 . Define t such that everyone in the first t agents maximizes the support utility at x_1 given the previous agents' locations, but this no longer holds for the $(t+1)$ -th agent.
 - When there are $k \leq t+1$ agent, we allocate them together at x_1 .
 - When there are $k \geq t+2$ agent, let $\vec{x} = (x, \dots, x)$ (the number of x is $k-2$). We define left largest support utility $ll(x)$ and right largest support utility $rl(x)$:

$$ll(x) = \max \left\{ \int_{z-0.25}^{z+0.25} \frac{f(y)}{c(y, \vec{x})} dy \mid z \leq x \right\}$$

$$rl(x) = \max \left\{ \int_{z-0.25}^{z+0.25} \frac{f(y)}{c(y, \vec{x})} dy \mid z \geq x \right\}$$

Here $c(y, \vec{x}) = k-2$ if $|y-x| \leq w/2$ and $c(y, \vec{x}) = 0$ if $|y-x| > w/2$. There exists x^* such that $ll(x^*) = rl(x^*)$. Let $x_l \leq x$ be a solution of z for $\int_{z-0.25}^{z+0.25} \frac{f(y)}{c(y)} dy = ll(x^*)$ and $x_r \geq x$ be a solution of z for $\int_{z-0.25}^{z+0.25} \frac{f(y)}{c(y)} dy = rl(x^*)$. We put the first $k-2$ agents at x^* , $(k-1)$ -th agent at x_l , k -th agent at x_r .

When there are $k \geq t+2$ agents in the second case, based on the width is one half, the union of the support of $(k-1)$ -th agent and k -th agent will cover the support of previous agents. In fact, the $(k-1)$ -th and k -th agents are the unique two winners.

THEOREM 5. *When $w = 0.5$, Algorithm 2 gives a Nash equilibrium.*

The main proof is omitted due to the space. We only prove that Algorithm 2 could find a location profile, i.e., there exists x^* such that $ll(x^*) = rl(x^*)$ when there are $k \geq t+2$ agents in the second case.

PROOF. When the first $k-2$ agents located at x_1 , suppose the support utility of the $(k-1)$ th agent is maximized at x_2 . In fact, we can prove $x_2 \neq x_1$. W.l.o.g, we assume $x_2 > x_1$, then $rl(x_1) \geq ll(x_1)$. Moreover

$$rl(0.75) = \int_{0.5}^1 \frac{f(y)}{k-1} dy \leq \int_{x_1-0.25}^{x_1+0.25} \frac{f(y)}{k-1} dy \leq ll(0.75)$$

Since $ll(x)$ is continuous and weakly increasing while $rl(x)$ is continuous and weakly decreasing, x^* exists. \square

For smaller width, the attraction interval may not overlap. This results new possibilities of the interval intersection relationship, and many possibilities about who is the winner. Hence the proof does not hold for smaller width.

THEOREM 6. *If there always exists a Nash equilibrium when $w = 0.5$, then it also holds for $w \geq 0.5$.*

In this case, the interval $(1-w, w)$ belongs to every agent's support. The idea is we can remove this interval and the problem becomes proving the existence of Nash equilibria with $w = 0.5$.

PROOF. When $w > 0.5$, we can construct a Nash equilibrium from an instance with $w = 0.5$. We let the new distribution function be

$$g(x) = \begin{cases} \frac{f((2-2w)x)}{\int_0^{1-w} f(y)dy + \int_w^1 f(y)dy} & x \leq 0.5 \\ \frac{f((2-2w)x + 2w - 1)}{\int_0^{1-w} f(y)dy + \int_w^1 f(y)dy} & x > 0.5 \end{cases}$$

Suppose there is a Nash equilibrium (x_1, x_2, \dots, x_n) under distribution g with width 0.5. We can verify that $((2-2w)x_1 + w - 0.5, (2-2w)x_2 + w - 0.5, \dots, (2-2w)x_n + w - 0.5)$ forms a Nash equilibrium in the distribution f and with width w . \square

5. FAIRNESS IN THE NASH EQUILIBRIUM

DEFINITION 4 (FAIRNESS). *Given a game G , define the fairness of the game to be:*

$$FAIR = \min_{\vec{x} \in NE(G)} \frac{\min_i u_i(\vec{x})}{\max_i u_i(\vec{x})}$$

Intuitively, given a location profile \vec{x} , the ratio $\frac{\min_i u_i(\vec{x})}{\max_i u_i(\vec{x})}$ describes how fairly the utilities are divided among all agents. We choose the lowest such ratio of Nash equilibria as our fairness criterion.

In the winner utility setting, The fairness is simply 0 if there exists a losing agent, or 1 otherwise. In the support utility setting, the fairness is generally not easy to compute. However, we give a tight lower bound in such a setting.

We first give a lemma that bounds the ratio $\frac{\min_i u_i(\vec{x})}{\max_i u_i(\vec{x})}$ for any Nash equilibrium.

LEMMA 2. *The utility of agent i is at least $\frac{1}{2 \lceil w_j/w_i \rceil}$ fraction of the utility of agent j . The bound is tight.*

PROOF. In the support utility maximization setting, we have

$$u_i \geq \int_s^{s+w_i} \frac{f(x)}{c(x, \vec{x}_{-i}) + 1} dx \geq \frac{1}{2} \int_s^{s+w_i} \frac{f(x)}{c(x, \vec{x}_{-i})} dx, \forall s.$$

The idea is to split the interval $(x_j - w_j/2, x_j + w_j/2)$ into many small intervals with size w_j , and apply the inequality on them:

$$\begin{aligned} \left\lceil \frac{w_j}{w_i} \right\rceil u_i &\geq \frac{1}{2} \sum_{k=1}^{\lceil \frac{w_j}{w_i} \rceil} \int_{x_i - \frac{w_j}{2} + w_j \cdot (k-1)}^{x_i - \frac{w_j}{2} + w_j \cdot k} \frac{f(x)}{c(x, \vec{x}_{-i})} dx \\ &\geq \frac{1}{2} \int_{x_i - \frac{w_j}{2}}^{x_i - \frac{w_j}{2} + \lceil \frac{w_j}{w_i} \rceil \cdot w_j} \frac{f(x)}{c(x, \vec{x}_{-i})} dx \geq \frac{u_j}{2} \end{aligned}$$

Consider the case distribution

$$f(x) = \begin{cases} 2/3 & x \leq 1/2 \\ 4/3 & 1/2 < x \end{cases}$$

There are two agents with the same width $1/2$, $(x_1 = 0.25, x_2 = 0.75)$ is a Nash equilibrium. We can see the ratio of support utility between two agent meets the bound $1/2$. \square

Suppose agent 1 has the largest support utility, agent n has the smallest support utility, we can easily get the ratio between the largest and smallest support utility is $\frac{1}{2 \lceil w_1/w_n \rceil}$.

The following theorem is immediate based on Lemma 2.

THEOREM 7. *The fairness in the support utility setting is at least $\frac{1}{2 \lceil w_M/w_m \rceil}$, where $w_M = \max\{w\}$ and $w_m = \min\{w\}$. The bound is tight.*

6. PRICE OF ANARCHY AND UPPER BOUND OF UNCOVERED SUPPORT

The price of anarchy is an important metric that measures how efficiency decreases due to agents' selfish behaviors. In particular, we define the price of anarchy as follows:

DEFINITION 5 (PRICE OF ANARCHY). *Given a game G , the price of anarchy of the game is*

$$PoA = \frac{\min_{\vec{x} \in NE(G)} \sum_{i=1}^n u_i(\vec{x})}{\max_{\vec{x}} \sum_{i=1}^n u_i(\vec{x})}$$

If we consider PoA in the winner utility maximization setting, the sum of the utility is always 1. There is no inefficiency. If we consider amount of uncovered support in the winner utility maximization setting, the upper bound could reach 1, which has a poor performance. To make the problem interesting, we mainly consider the support utility maximization setting.

First, consider the price of anarchy.

THEOREM 8. *The price of anarchy of the support utility maximization is at least $\frac{1}{2}$. The bound is tight.*

PROOF. Suppose the optimal location profile that maximizes the sum of support utilities is $\vec{x}^* = (x_1^*, x_2^*, \dots, x_n^*)$, and the Nash equilibrium location profile is (x_1, x_2, \dots, x_n) . The sum of support utilities in \vec{x}^* is upper bounded by adding n agents with location \vec{x} ,

$$\begin{aligned} \sum_{i=1}^n u_i(\vec{x}^*) &\leq \sum_{k=1}^{2n} u_k(\vec{x}^*, \vec{x}) < \sum_{k=1}^n u_k(x_i^*, \vec{x}_{-i}) + \sum_{k=1}^n u_k(x_i, \vec{x}_{-i}) \\ &\leq \sum_{k=1}^n u_k(x_i, \vec{x}_{-i}) + \sum_{k=1}^n u_k(x_i, \vec{x}_{-i}) = 2 \sum_{k=1}^n u_k(\vec{x}) \end{aligned}$$

Thus, $PoA \geq 1/2$. \square

Actually, when n goes to infinity, the PoA can be arbitrarily close to $1/2$. Consider the example, there are n agents with the same width $1/n$.

$$f(x) = \begin{cases} \frac{n^2}{2n-1} & x \in [0, 1/n] \\ \frac{n}{2n-1} & x \in (1/n, 1] \end{cases}$$

The optimal location profile is $(\frac{1}{2n}, \frac{3}{2n}, \dots, \frac{2n-1}{2n})$, i.e., the union of the support covers the $[0, 1]$ interval. The optimal support utility is 1. While, consider the Nash equilibrium $(\frac{1}{2n}, \frac{1}{2n}, \dots, \frac{1}{2n})$, i.e., all the agents are located at point $\frac{1}{2n}$. The support utility in this Nash equilibrium is $\frac{n}{2n-1}$. When n goes to infinity, the PoA converges to $1/2$.

Then we can consider how many clients are not served.

THEOREM 9. *The support of the uncovered clients is at most $\frac{1}{1 + \sum_i w_i}$*

PROOF. We let p denote the "uncovered support", q denote "covered support". Suppose $\frac{u_1}{w_1} = \min\{\frac{u_i}{w_i}\}$, i.e., agent 1 has the lowest density of the support utility. Then the sum of all agents' support utility is at least

$$q \geq \sum_i w_i \cdot \frac{u_1}{w_1}$$

We split the interval $[0, 1]$ into pieces with size w_1 . If we don't count agent 1, then in each small pieces, the support of the uncovered set is at most u_1 . Otherwise, agent 1 will deviate to cover this interval. Then the sum of the uncovered support is at most $\lceil \frac{1}{w_1} \rceil \cdot u_1$. Since agent 1 has covered u_1 , then actually the uncovered support can be limited,

$$p \leq \left(\left\lceil \frac{1}{w_1} \right\rceil - 1 \right) \cdot u_1 \leq \frac{u_1}{w_1}$$

At last we have

$$\begin{aligned} p &= \frac{p}{p+q} = \frac{1}{1+q/p} \\ &\leq \frac{1}{1 + (\sum_i w_i \cdot \frac{u_1}{w_1}) / (\frac{u_1}{w_1})} = \frac{1}{1 + \sum_i w_i} \end{aligned}$$

\square

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