Multiplayer General Lotto game

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Abstract. In this paper, we investigate the multiplayer General Lotto game – a significant extension of the classic Colonel Blotto game – in a setting involving competition across multiple battlefields. Under this framework, resources are allocated probabilistically by players, ensuring their expected expenditures remain within individual budgets. Our contributions begin by establishing the existence of Nash equilibrium for general scenarios, accommodating asymmetries in both player budgets and battlefield valuations. A detailed characterization of equilibrium strategies follows, specifically addressing the complexities arising when multiple players compete on a single battlefield, and culminating in a system of equations for computing equilibria. Furthermore, we identify conditions under which equilibrium uniqueness is guaranteed in single-battlefield game. Turning to multi-battlefield competition, our analysis reveals an upper bound on the average number of battlefields actively contested by each player. For symmetric scenarios, we provide explicit equilibrium solutions. Finally, equilibrium multiplicity is demonstrated concretely through an illustrative example involving multiple players and battlefields.

Keywords: Colonel Blotto game · Nash Equilibrium · General Lotto game.

1 Introduction

The Colonel Blotto game (CB game) is one of the simplest and most well-known game-theoretic models for resource allocation. Initially proposed by Borel [5], the CB game has been widely studied over the years [6]. In this game, two competitors, A and B, are each given a fixed budget of resources to compete over n battlefields. Each player assigns a specific value to each battlefield and simultaneously allocates their resources across these n battlefields, ensuring that the total allocation does not exceed their respective budgets. On each battlefield, the player with the highest allocation wins and receives the value assigned to that battlefield, while the losing player receives no reward. Consequently, each player's utility is the sum of their gains from all n battlefields. The objective for each player is to maximize their own utility by strategically choosing their resource distribution.

In addition to the standard CB game, several variants have been proposed [3, 4]. One of the most well-known is the General Lotto (GL) game, where the budget constraint is relaxed so that aggregate allocations do not exceed a player's budget only in expectation, rather than with certainty [13, 12, 19, 24, 8, 2, 27]. The GL game models scenarios in which multiple items are allocated through repeated, independent all-pay auctions, and several budget-constrained bidders compete for these items on a daily basis. For example, consider the case where several machine learning competitions start online every day, each with a new task. Multiple entities, such as academic research groups or companies, compete for the first place in each competition. These entities may value each competition's first place differently. Each entity has a fixed amount of computing power, determining the total computation they can perform each day. For every competition, the entity that spends the most computation on its task wins. To increase the probability of winning a competition, an entity must allocate more computation to it, at the cost of either (i) having less computation to spend on other competitions, or (ii) performing poorly in competitions that start in the following days (assuming each competition has a relatively long duration, allowing entities to accumulate several days' worth of computation for a single competition). In fact, as demonstrated in the proof of Theorem 1, there exists a threshold amount of computation beyond which no player finds it beneficial to accumulate more for any single competition.

To our knowledge, in non-cooperative game environment, current research on the GL game with asymmetric budgets has been limited to the two-player case [13, 12, 19, 24, 29]. However, many real-world scenarios involve multiple participants, such as market competition, international relations, social networks, and ecosystems. These situations all involve strategic resource allocation and competition, and a GL game framework with multiple players can more accurately model interactions. Additionally, in applications such as auction design and public policy making, designers aim to guide participants toward a specific equilibrium by adjusting rules or mechanisms. Research on Nash equilibrium in multiplayer settings can aid in identifying and designing effective mechanisms for these purposes.

Investigating the existence, uniqueness, and structure of Nash equilibrium in multiplayer settings is an important but highly challenging task. In the context of two players, Roberson and Kovenock [19] completely characterized the Nash equilibria in a general setting in which players have asymmetric budgets and the battlefield values are also asymmetric between players. In this paper, we extend their setting from two players to multiple players. However, analyzing the Nash equilibrium in multiplayer settings introduces certain technical challenges. These challenges mainly fall into the following three aspects:

1. In the case of a single battlefield, Roberson and Kovenock [19] showed that when the GL game involves only two players, there exists a closed-form solution for the Nash equilibrium. When there are only two players, their supports are identical in Nash equilibrium. However, when the number of players in the GL game exceeds two, such a closed-form solution no longer exists. We discover that the upper endpoints of the supports of players' strategies coincide, and that the minimum value of a player's support above zero inversely correlates with his budget in Nash equilibrium.
2. In the case of multiple battlefields, Roberson and Kovenock [19] proved that the GL game has a unique set of Nash equilibrium univariate marginals. They used a parameter related to players' budgets to partition the battlefields into regions where each player has an advantage. However, in the multi-player, multi-battlefield setting, the set of Nash equilibrium univariate marginals is not unique. The multiplicity of Nash equilibria makes analyzing Nash equilibria more challenging.
3. Regarding the existence of Nash equilibrium, Roberson and Kovenock [19] employed a constructive proof approach. However, when multiple players participate in the GL game, this constructive method becomes inapplicable due to the complicated computation of Nash equilibrium. We use a "discretization + limiting" game framework, applying Kakutani's fixed point theorem and Helly's selection theorem to prove the existence of Nash equilibria.

1.1 Our Contribution

In this work, we focus on the multiplayer GL game with asymmetric budgets over multiple heterogeneous battlefields. Our key contributions are as follows:

- We establish the existence of a Nash equilibrium in the GL game. We start with constructing a variant of the GL game where each player's bid space is discrete and bounded from above, and we demonstrate the existence of a Nash equilibrium in this modified game. Subsequently, we show that if the threshold is sufficiently large, it becomes non-restrictive. A Nash equilibrium in the GL game arises from the limit of a sequence of Nash equilibria in the modified games, where the bid grid in the sequence becomes finer and finer. A more detailed discussion of the approach is given at the end of Section 3.
- For the game with a single battlefield, we provide a comprehensive characterization of Nash equilibrium, revealing a relationship between the relative order of players' budgets and the support of their strategies. This characterization also naturally implies the known result for the Nash equilibrium in the case of two players with asymmetric budgets [19]. Additionally, we provide a system of equations to solve for the Nash equilibria. Moreover, we prove the uniqueness of Nash equilibrium when there are at least two players with the maximum budget.
- For the game with multiple battlefields, we prove that for almost every value profile, each player focuses only on few battlefields when the number of players is sufficiently large. We show that Nash equilibrium is not unique by

providing an example. Additionally, in the symmetric setting with multiple players and multiple battlefields, we present a solution for the Nash equilibrium.

To our knowledge, this is the first paper to study the multiplayer GL game with asymmetric budgets. Our results significantly extend the existing literature on the GL game.

Due to space limitations, some lemmas and most of the proofs are included in the full version of the paper [4].

1.2 Related Work

The CB game was initially developed to simulate military logistics, where resources are analogous to soldiers, equipment, or weapons. Owing to its ability to model a variety of real-world scenarios, the CB game has aroused considerable interest among scholars in fields such as sociology, mathematics, economics, and computer science [28, 17, 15, 11, 23, 26, 18, 29, 20]. In the CB game, there may not be pure strategy Nash equilibrium; however a mixed strategy Nash equilibrium does exist, represented by a pair of n-variate distributions [26, 25, 22, 21]. Research on the CB game primarily focuses on identifying and computing Nash equilibria [26, 14, 1, 30]. However, the computation of Nash equilibrium in this game is nontrivial, because the equilibrium strategies correspond to complicated joint distributions defined on an n-dimensional simplex [26, 10, 16, 4, 14].

The GL game, as the most well-known variant of the CB game, has also been extensively studied. Beale and Heselden [2] were the first to introduce the GL game, proposing it as an auxiliary construct for deriving approximate solutions to the Colonel Blotto game. Paarporn et al. [24] considered the GL game under an information asymmetry setting, where one player's budget was public knowledge while the other player's budget was drawn from a Bernoulli distribution. Chandan et al. [8] studied the GL game with a concession, where the players' concession strategies enables players to reach a more effective Nash equilibrium in a competitive environment. Chandan et al. [9] investigated how, in the GL game, openly declaring strategic intentions can influence an opponent's strategy choices, helping players to secure better outcomes in specific situations and gain an advantage. In addition, Chandan et al. [7] explored the optimal strategies for resource allocation in a multi-stage GL game. Their work indicated that by dynamically adjusting investment strategies, players can significantly increase their winning rates at different stages, thereby maximizing overall returns.

2 Preliminaries

In the multiplayer General Lotto (GL) game, a set of n players, indexed by $[n] := \{1, 2, \cdots, n\}$, compete across m battlefields, indexed by $[m] := \{1, 2, \cdots, m\}$. Each player $i \in [n]$ is equipped with a fixed budget $B_i > 0$ and a valuation

[4] The full paper is available at `https://arxiv.org/abs/2401.14613`.

$v_{i,j} > 0$ for each battlefield $j \in [m]$. A player's *pure strategy* is to distribute their budget, referred to as *bids*, across the battlefields. The outcome on each battlefield is determined independently; the player who places the highest bid on a battlefield wins that battlefield and gains the associated value. All other players receive no reward for that battlefield. In cases where multiple players tie for the highest bid, the winner is chosen uniformly at random.

Following the model introduced by Kovenock and Roberson [19], we consider mixed strategies in which a player's bid on each battlefield is modeled as a nonnegative random variable. That is, instead of choosing fixed bids, player i selects a tuple $(F_{i,j})_{j\in[m]}$, where each $F_{i,j}$ is a cumulative distribution function (c.d.f.) from which their actual bid, denoted $X_{i,j}$, is drawn. The total expected bid across all battlefields must respect the budget constraint. The set of feasible strategies $\mathcal{F}_i$ includes all such m-tuples $(F_{i,j})_{j\in[m]}$ that meet the following conditions:
(i) the support of each $F_{i,j}$, denoted by $Supp_{i,j}$, is contained in $[0, +\infty)$, ensuring all bids are nonnegative, and
(ii) the total expected bid across all battlefields does not exceed player i's budget B_i.
We therefore represent the set of feasible strategies for player i as

$$\mathcal{F}_i := \left\{(F_{i,j})_{j\in[m]}: \quad \sum\nolimits_{j=1}^{m} \mathbb{E}_{X_{i,j}\sim F_{i,j}}\left[X_{i,j}\right] \le B_i, \quad Supp_{i,j} \subseteq [0, +\infty)\right\}. \tag{1}$$

We use $F_{i,j}(x^+)$ and $F_{i,j}(x^-)$ to denote the right-hand and left-hand limits of the distribution function $F_{i,j}$ at point x, respectively. Throughout the paper, the notation $-i$ refers to all players except player i. On each battlefield j, the bids $(X_{i,j})_{i\in[n]}$ are independently distributed. Specifically, suppose the realization of the random variable $X_{i,j}$ takes the value $x \ge 0$, while the bids of all other players on battlefield j, $(X_{i',j})_{i'\neq i}$, are drawn independently from their respective distributions $F_{-i,j}$. Then, the probability that player i wins battlefield j is given by

$$\Pr[i \text{ wins } j \text{ by bidding } x] = \mathop{\mathbb{E}}_{X_{-i,j}\sim F_{-i,j}}\left[\frac{\mathbb{I}[x \ge X_{i',j}, \forall i' \neq i]}{\#\{i' \neq i : X_{i',j} = x\} + 1}\right], \tag{2}$$

where $\mathbb{I}[\cdot]$ is the indicator function, equal to 1 if the predicate is true and 0 otherwise, and $\#\{\cdot\}$ denotes the cardinality of a set, i.e., the number of elements in the set.

Player i's expected utility on battlefield j is given by their valuation v_{ij} multiplied by the probability of winning that battlefield. Their total expected utility is the sum of expected utilities across all battlefields. These are formally defined as:

$$u_{i,j}(x, F_{-i,j}) = v_{ij} \cdot \mathop{\mathbb{E}}_{X_{-i,j}\sim F_{-i,j}}\left[\frac{\mathbb{I}[x \ge \max_{i'\in[n]} X_{i',j}]}{\#\{i' \neq i : X_{i',j} \ge x\} + 1}\right].$$
$$U_i\left((F_{i,j})_{j\in[m]}, (F_{-i,j})_{j\in[m]}\right) = \sum\nolimits_{j\in[m]} \mathop{\mathbb{E}}_{X_{i,j}\sim F_{i,j}}\left[u_{i,j}(X_{i,j}, F_{-i,j})\right].$$

Players are utility maximizers. Given the strategy profile of the other players $(F_{-i,j})_{j\in[m]}$, player i's best response is a strategy that maximizes their expected utility:

$$(F_{i,j})_{j\in[m]} \in \underset{(F'_{i,j})_{j\in[m]}\in\mathcal{F}_i}{\arg\max}\; U_i\left((F'_{i,j})_{j\in[m]}, (F_{-i,j})_{j\in[m]}\right).$$

A strategy profile $(F_{i,j})_{i\in[n],j\in[m]}$ is a Nash equilibrium if, for every player i, their strategy is a best response to the strategies of all other players.

Throughout the paper, we focus on the mixed strategies of the players, i.e., $F_{i,j}$, $\forall i \in [n]$, $\forall j \in [m]$.

The following lemma provides a necessary condition for a strategy profile to be a Nash equilibrium. Intuitively, on each battlefield, a player's utility must be bounded above by a linear function of their own bid. Moreover, the player assigns positive probability only to those bids where this upper bound is tight. That is, the utility equals the value of the linear function. Otherwise, the player could deviate to a different bid that yields strictly higher utility without altering the expected allocation to that battlefield. Additionally, for each player, the slopes of these linear upper bounds must be the same across all battlefields. If they differed, the player could improve their utility by shifting expected allocations between battlefields.

Lemma 1. *If $(F_{i,j})_{i\in[n],j\in[m]}$ is a Nash equilibrium, then there exist constants $a_i > 0$ for every $i \in [n]$ and $b_{i,j} \geq 0$ for every $i \in [n], j \in [m]$ such that*

$$\Pr_{X_{i,j}\sim F_{i,j}}\left[u_{i,j}(X_{i,j}, F_{-i,j}) = a_i X_{i,j} + b_{i,j}\right] = 1,$$

and for all $x \geq 0$, it holds that

$$u_{i,j}(x, F_{-i,j}) \leq a_i x + b_{i,j},$$

for every $i \in [n]$ and $j \in [m]$.

We next establish a property of the constants $b_{i,j}$ that further simplifies the analysis.

Lemma 2. *For every battlefield $j \in [m]$, at most one player can have $b_{i,j} > 0$. That is, $\#\{i : b_{i,j} > 0\} \leq 1$.*

This result will be particularly useful in our later analysis, especially when characterizing Nash equilibria on a single battlefield.

3 Existence of Nash Equilibrium in the General Lotto game

In this section we show that the GL game has a Nash equilibrium.

Theorem 1. *A Nash equilibrium exists in the GL game.*

The proof of Theorem 1 relies on the following game $\mathcal{G}_k$, which is a modification of the GL game where only bounded bids on discrete grids are allowed, and ties are broken uniformly at random.

For every $k = 1, 2, \ldots$, consider the following game $\mathcal{G}_k$:

- The finite set of feasible bids is given by $A_k := \{\frac{l}{k} \cdot T \mid 0 \le l \le k, l \in \mathbb{Z}\}$, where $T = 2^{2n+3} \cdot \max_{i\in[n]} B_i$.
- Every player $i \in [n]$ chooses a strategy, which is given by m distributions $F_{i,\cdot,k} := (F_{i,j,k})_{j\in[m]}$ with every $F_{i,j,k} \in \Delta(A_k)$, so that player i's random bid on battlefield j is given by $X_{i,j} \sim F_{i,j,k}$, satisfying constraint $\sum_{j\in[m]} \mathbb{E}[X_{i,j}] \le B_i$. Note that the bids $(X_{i,j})_{i\in[n],j\in[m]}$ are independently distributed.
- Denote the strategies of all players other than i on battlefield j by $F_{-i,j,k} := (F_{i',j,k})_{i'\ne i}$, and the random bids of all players other than i on battlefield j by $X_{-i,j} := (X_{i',j})_{i'\ne i}$. Denote the strategy profile by $F_{\cdot,\cdot,k} := (F_{i,\cdot,k})_{i\in[n]}$, and the strategies of players other than i by $F_{-i,\cdot,k}$.
- Denote the set of i's feasible strategies by $\mathcal{F}_{i,k}$, which is a compact subset of $(\Delta(A_k))^m$. Denote the set of feasible strategy profiles by $\mathcal{F}_k := \times_{i\in[n]}\mathcal{F}_{i,k}$.
- Break ties uniformly at random: For each $i \in [n]$ and each $F_{\cdot,\cdot,k} \in \mathcal{F}_k$, the utility of player i's is given by the function $U_i : \mathcal{F}_k \to \mathbb{R}$, defined as:

$$u_{i,j}(x, F_{-i,j,k}) = \mathop{\mathbb{E}}_{X_{-i,j}\sim F_{-i,j,k}} \left[\frac{v_{ij} \cdot \mathbb{I}[x \ge \max_{i'\in[n]} X_{i',j}]}{\#\{i' \ne i : X_{i',j} \ge x\} + 1} \right].$$

$$U_i(F_{\cdot,\cdot,k}) = \sum_{j\in[m]} \mathbb{E}_{X_{i,j}\sim F_{i,j,k}} \left[u_{i,j}(X_{i,j}, F_{-i,j,k}) \right].$$

Throughout this section, given a positive integer k, we denote a Nash equilibrium of $\mathcal{G}_k$ by $\tilde{F}_{\cdot,\cdot,k}$ and represent a Nash equilibrium strategy of player i by $(\tilde{F}_{i,\cdot,k})_{j\in[m]}$.

Based on our constructed discrete game $\mathcal{G}_k$, we adopt the following steps to prove the existence of Nash equilibrium in the GL game:

- Show that $\mathcal{G}_k$ admits a Nash equilibrium.
- Show that no player bids in $[\frac{T}{2} + \frac{T}{k}, T]$ when k is sufficiently large.
- Show that $\tilde{F}_{i,j,k}$ converges when $k \to \infty$.
- Show that $\tilde{F}_{i,j,k}$ is uniformly continuous in the interval $[0, T]$ when $k \to \infty$.
- Show that the existence of a Nash equilibrium in $\mathcal{G}_k$ can be extended to the original game GL.

One can apply Kakutani's fixed point theorem to a proper set-valued function $\beta : \mathcal{F}_k \to 2^{\mathcal{F}_k}$ and conclude that a Nash equilibrium exists in $\mathcal{G}_k$.

Lemma 3. *For every $k \ge 1$, $\mathcal{G}_k$ has a Nash equilibrium.*

It should be noted that Nash's theorem cannot be used to prove Lemma 3. Under Nash's theorem, each player's feasible mixed strategy set is a simplex, with each vertex corresponding to a pure strategy, and every pure strategy for a player must not exceed their budget. However, in our model, each player's strategy must satisfy the constraint that the expected total allocation does not exceed their budget. This means that, on battlefield j, player i's mixed strategy may assign positive probability to bids that individually exceed their budget.

For any $\tilde{F}_{\cdot,\cdot,k}$, Lemma 4 shows that the supports of the Nash equilibrium strategies are uniformly bounded away from the upper threshold T. This means that, as long as T is chosen sufficiently large relative to the players' budgets, the actual Nash equilibrium strategies are unaffected by the specific value of T. The intuition is as follows: if a player bids a sufficiently high value, such as $3\cdot\max_{i\in[n]} B_i$, then by Markov's inequality and the expected budget constraints, the probability that any other player's bid exceeds this amount must be less than $\frac{1}{2}$. Thus, a player who bids $3\cdot\max_{i\in[n]} B_i$ can beat any single opponent with probability greater than $\frac{1}{2}$, and wins against all opponents simultaneously with probability greater than $(\frac{1}{2})^{n-1}$. However, such a high bid is very costly in expectation and not optimal under the budget constraint. Therefore, in Nash equilibrium, players will not put probability mass near the upper threshold, and the strategies concentrate on much lower bids.

Lemma 4. *Given $T = 2^{2n+3}\cdot\max_{i\in[n]} B_i$, there exists $K > 0$ s.t. for any $k \geq K$, no player bids in $[\frac{T}{2}+\frac{T}{k}, T]$ on any battlefield j in Nash equilibrium of $\mathcal{G}_k$.*

Next, we show that the sequence $\{F_{i,j,k}\}_{k\geq 1}$ admits a subsequence that converges to a limit distribution when $k\to\infty$. Because every distribution function $F_{i,j,k}$ is monotone and bounded in $[0,1]$, the family $\{F_{i,j,k}\}_{k\geq 1}$ forms a set of bounded monotone functions. Helly's selection theorem guarantees that for any fixed coordinate (i,j), one can extract a pointwise-convergent subsequence. Since the numbers of players n and battlefields m are finite, we iterate over the coordinates $(1,1),(1,2),\ldots,(n,m)$: first take a convergent subsequence for $(1,1)$; within that subsequence, extract another for $(1,2)$; and so on. After finitely many steps we obtain an infinite subsequence $\{k_\ell\}$ with $k_\ell\to\infty$ along which $F_{i,j,k_\ell}(x)$ converges for every x and for every (i,j). Thus, the limits $F_{i,j}(x) = \lim_{\ell\to\infty} F_{i,j,k_\ell}(x)$ define the desired limit distributions. Hence, using Helly's "coordinate-by-coordinate, nested subsequence" procedure and the fact that the total number of coordinates is finite, we ensure that the full strategy sequence converges simultaneously along a common subsequence.

In the remaining part of this section, we refine $(\tilde{F}_{\cdot,\cdot,k})_{k\geq 1}$ to be the final subsequence obtained after the above steps, in which every $(\tilde{F}_{i,j,k})_{k\geq 1}$ converges pointwise. Denote the final limit of the sequence as $\tilde{F}_{\cdot,\cdot} := (\tilde{F}_{i,j})_{i\in[n],j\in[m]}$. Note that for $\tilde{F}_{i,\cdot,k}$ to be a best response to $\tilde{F}_{-i,\cdot,k}$, it must satisfy the condition $\sum_{j\in[m]}\mathbb{E}_{X_{i,j}\sim\tilde{F}_{i,j,k}}[X_{i,j}] = B_i$, which in turn implies that $\sum_{j\in[m]}\mathbb{E}_{X_{i,j}\sim\tilde{F}_{i,j}}[X_{i,j}] = B_i$.

Note that although every c.d.f. $\tilde{F}_{i,j,k}$ is right-continuous, the limit $\tilde{F}_{i,j}$ does not necessarily have continuity. By taking some sufficiently large k and analyzing Nash equilibrium of $\mathcal{G}_k$, we can establish some properties about the continuity of $\tilde{F}_{\cdot,\cdot}$ in $[0,T)$ (See Lemmas 13 and 14 in Appendix B of the full version). Based on the continuity, we can derive Lemma 5, which establishes that the convergence of $(\tilde{F}_{\cdot,\cdot,k})_{k\geq 1}$ is uniform and furthermore ensures that the sequences $u_{i,j}$ also converge.

Lemma 5. *Given $n \geq 3$, for every $i \in [n], j \in [m]$, we have:*

1. *$\tilde{F}_{i,j}$ is uniformly continuous on $[0,T]$,*
2. *the sequence $(\tilde{F}_{i,j,k})_{k\geq 1}$ uniformly converges to $\tilde{F}_{i,j}$,*
3. *the sequence of univariate functions $u_{i,j}(\cdot, \tilde{F}_{-i,j,k})$ will converge in $||\cdot||_\infty$ to $u_{i,j}(\cdot, \tilde{F}_{-i,j})$, where $u_{i,j}(\cdot, \tilde{F}_{-i,j})$ is the utility function of original GL game with continuous bid space. That is,*

$$\lim_{k\to\infty} \sup_{x\in[0,T]} |u_{i,j}(x, \tilde{F}_{-i,j,k}) - u_{i,j}(x, \tilde{F}_{-i,j})| = 0.$$

Lemma 5 leads to the conclusion that $\tilde{F}_{\cdot,\cdot}$ is a Nash equilibrium in the game where the players can choose random bids within interval $[0,T]$. By Lemma 4, we can know that no player bids in $[\frac{3T}{4}, T]$. In fact, even if players are allowed to bid more than T, in Nash equilibrium, no player will actually bid above T. Thus, we can establish the existence of a Nash equilibrium in the GL game. Due to space limitations, the proof of Theorem 1 can be found in Appendix B of the full version.

Remark on the proof approach. In the General Lotto game, the existence of a Nash equilibrium cannot be established directly by a fixed-point theorem. In particular, even a more powerful variant of the Kakutani theorem, namely the Fan–Glicksberg theorem, is not applicable. The reasons are as follows: To prove the existence result using a fixed-point theorem, we need to consider players' best responses. Let the players' best response be represented by a correspondence $\Gamma : F \rightrightarrows F$. The correspondence $\Gamma_i : F_{-i} \rightrightarrows F_i$ is required to satisfy the following properties: first, Γ_i must be compact, convex, and nonempty; second, Γ_i must be upper hemicontinuous (u.h.c.). However, in our game, neither of these properties holds naturally. For the first property, because of the tie-breaking rule, a best response may fail to exist. For instance, assuming there are only two players competing, if player 1 bids 1 with certainty, then there is no best response for player 2. In addition, since the action space is unbounded, the value of Γ_i may also be unbounded. For the second property, because of the tie-breaking rule, $\Gamma_i(\lim_{F_{-i}\to F^*_{-i}} F_{-i})$ may fail to exist, where F^*_{-i} denotes the limit of the sequence F_{-i}. Even if it exists, the utility function is not continuous in the presence of ties; hence the best response mapping fails to be upper hemicontinuous. These obstacles make a direct application of the fixed-point theorem impossible, which explains the technical necessity of our discretization-based approach.

4 Nash Equilibrium Characterization of A Single Battlefield

In this section, we analyze Nash equilibrium in the case of a single battlefield, i.e., the special case where $m = 1$. We begin by providing a complete characterization of the structure of Nash equilibrium. Our analysis reveals that the upper endpoints of the supports of the players' Nash equilibrium strategies coincide, and the minimum value of a player's support above zero is inversely correlated with their budget. Furthermore, we prove the uniqueness of the Nash equilibrium when at least two players share the maximum budget.

4.1 Characterization of Nash Equilibrium

With only one battlefield, we omit the subscript j in this section, and any Nash equilibrium can be represented by a distribution profile $(F_i)_{i\in[n]}$ on the single battlefield. Additionally, without loss of generality, we assume $v_i = 1$ for every player i.

We first show that for any Nash equilibrium, $\forall i \in [n]$, F_i has no mass point in $(0, +\infty)$ (see Lemma 6 and Lemma 7). Then we establish that, in any Nash equilibrium, 0 is the common infimum of the supports of all players' strategies, and the common supremum of the supports of all players' strategies is the same (see Lemma 8 and Lemma 9).

The following lemma states that, for any $x > 0$, in any Nash equilibrium, there can be at most one player i whose strategy $F_i(x)$ assigns a non-zero measure at x.

Lemma 6. *For any Nash equilibrium $(F_i)_{i\in[n]}$, we have $\#\{i : F_i(x) \neq F_i(x^-)\} \leq 1$, $\forall x > 0$.*

The next lemma examines the characteristics of the support of players' strategies in a Nash equilibrium.

Lemma 7. *For any Nash equilibrium $(F_i)_{i\in[n]}$, we have*

1. *$\bigcup_i Supp_i$ is an interval starting from 0.*
2. *$\bigcup_{i'\neq i} Supp_{i'} = \bigcup_{\hat{i}} Supp_{\hat{i}}$, $\forall i$.*
3. *$\forall x \in \bigcup_i Supp_i$, $\#\{i | x \in Supp_i\} \geq 2$.*

Our next lemma states that, for any player, the support of their strategy in a Nash equilibrium either includes the point $\{0\}$ or contains points arbitrarily close to 0.

Lemma 8. *Let $\epsilon > 0$, which can be chosen to be arbitrarily small. For any Nash equilibrium $(F_i)_{i\in[n]}$, $\forall i$, we have $[0, \varepsilon] \cap Supp_i \neq \emptyset$.*

The following lemma is key to the proof of Theorem 2. It asserts that the support of each equilibrium strategy, excluding $\{0\}$, is a single continuous interval, and such continuous intervals of all players share the same right endpoint.

Lemma 9. *For any Nash equilibrium $(F_i)_{i\in[n]}$, $\forall i$, we have that there exists $L > 0$ such that $\sup Supp_i = L$, and $Supp_i \cap (0, L) = (c_i, L)$, for some $c_i \geq 0$.*

By Lemma 9, let L denote the common supremum of all players' supports, that is, $L = \sup Supp_i$ for every player i. Using Lemmas 6 to 9, we establish a complete characterization of the structure of Nash equilibrium. Theorem 2 states that when there is a single player with the largest budget, the support of this player's equilibrium strategy is a single continuous interval $[0, L]$. For the player with the second largest budget, the support of its equilibrium strategy is also a single continuous interval $[0, L]$, but it includes a non-zero measure at 0. For all other players, the supports of their equilibrium strategies all include a non-zero measure at 0, and are increasing in their budgets in subset relationship, as illustrated in the plot on the left-hand side of Figure 1. When multiple players have the largest budget, the supports of their equilibrium strategies are all $[0, L]$. For the remaining players, the supports of their equilibrium strategies include a non-zero measure at 0, and the supports are increasing in their budgets in subset relationship, as depicted in the plot on the right-hand side of the Figure 1.

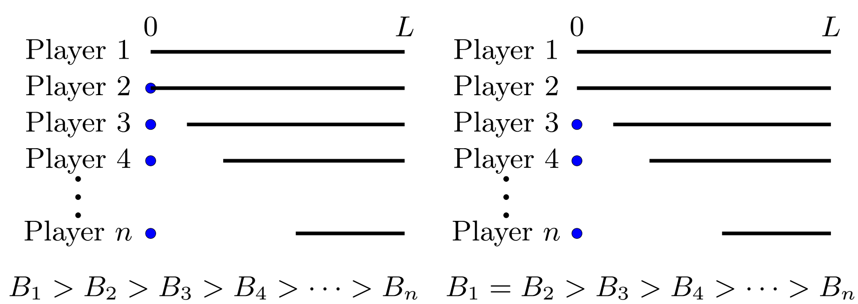

Fig. 1. The support of Nash equilibrium strategies, blue dot indicates mass point of distribution, black line represents support of distribution. The only difference between the left and right figures is that, in the left figure, player 2 has a mass point at 0, whereas in the right figure, player 2 does not have a mass point at 0.

Theorem 2. *Relabel the players so that $B_1 \geq B_2 \geq \cdots \geq B_n$, and define $i' := \max\{i : B_i = B_2\}$. For any Nash equilibrium $(F_i)_{i\in[n]}$ and supports $(Supp_i)_{i\in[n]}$ of equilibrium strategies, there exists $L > 0$ such that:*

1. *If $B_1 > B_2$, then we have*
   (a) $Supp_1 = [0, L]$, $F_1(0) = 0$,
   (b) $Supp_i = [0, L]$ and $F_i(0) > 0$ for all $i \in \{2, 3, \cdots, i'\}$,
   (c) $Supp_i = \{0\} \cup [h_i, L]$ and $F_i(0) > 0$, with $h_{i'+1} > 0$ and $h_i \geq h_{i-1}$, for all $i \geq i' + 1$.
2. *If $B_1 = B_2$, then we have*
   (a) $Supp_i = [0, L]$ and $F_i(0) = 0$ for all $i \in \{1, 2, \cdots, i'\}$.
   (b) $Supp_i = \{0\} \cup [h_i, L]$ and $F_i(0) > 0$, with $h_{i'+1} > 0$ and $h_i \geq h_{i-1}$, for all $i \geq i' + 1$.

Theorem 2 provides the structure of the support of Nash equilibrium strategies for multiple players on a single battlefield. These structures are crucial for determining Nash equilibrium strategies. Based on these structures, we can deduce the relationships between the strategies of the players. We find that, whether $B_1 > B_2$ or $B_1 = B_2$, the strategy of player 2 is closely related to the strategies of the subsequent players. The following lemma presents two important properties: (1) the strategy of any player $i \in \{3, 4, \cdots, n\}$ is identical to that of player 2 on player i's support set, excluding $\{0\}$, and (2) player 1 and player 2 adopt the same strategy if $B_1 = B_2$.

Lemma 10. *Given the budget vector $\boldsymbol{B} = (B_1, B_2, \cdots, B_n)$ satisfies $B_1 \geq B_2 \geq B_3 \geq \cdots \geq B_n$, then we have that for any player $i \in \{3, 4, \cdots, n\}$, $F_i(x) = F_2(x)$ for all $x \in Supp_i \backslash \{0\}$. Additionally, if $B_1 = B_2$, then $F_1(x) = F_2(x)$ for all $x \in Supp_2$.*

Let p_i denote the probability of the player i bidding 0 and $h_i = \inf(Supp_i \backslash \{0\})$. The following lemma describes the relationship between p_i and h_i for player $i \in \{3, 4, \cdots, n\}$.

Lemma 11. *For player $i \in \{3, 4, \cdots, n\}$, $F_i(h_i) = F_2(h_i) = p_i$.*

According to Theorem 2, Lemma 10 and Lemma 11, we can observe that in Nash equilibrium, the strategy of player $i \in [n]$ can be expressed by F_1, F_2, h_i, L. We can set up a system of equations to solve for the Nash equilibrium. The system can be divided into two cases:

- Case (1): $B_1 > B_2 \geq B_3 \geq \cdots \geq B_n$.
- Case (2): $B_1 = B_2 \geq B_3 \geq \cdots \geq B_n$.

Owing to space constraints, we only present the system of equations corresponding to the Case (2), while the system for Case (1) is included in the Appendix C of the full version.

Define $Q_i = \prod_{r>i} p_r$. For Case (2), we have $B_1 = B_2 = \cdots = B_{i'} > B_{i'+1} \geq \cdots \geq B_n$, which implies that $0 < h_{i'+1} \leq \cdots \leq h_n$, $0 = h_1 = \cdots = h_{i'}$, and $p_i > 0$, $\forall i \geq i' + 1$. Therefore, the system of equations is as follows:

$$\forall x \in [h_n, L], \forall i \in [n], \ \begin{cases} F_i(x) = (\frac{x}{L})^{\frac{1}{n-1}}, \\ \int_{h_n}^{L} x f_n(x) dx = B_n. \end{cases}$$

$$\forall x \in [h_r, h_{r+1}], \forall r \in \{i'+1, \cdots, n-1\}, \forall i \leq r, \ \begin{cases} F_i(x) = \left[\frac{x}{LQ_r}\right]^{\frac{1}{r-1}}, \\ \int_{h_r}^{h_{r+1}} x f_r(x) dx = B_r - B_{r+1}. \end{cases}$$

$$\forall x \in [0, h_{i'+1}], \forall r \in \{1, 2, \cdots, i'\}, \ \begin{cases} [F_r(x)]^{i'-1} Q_{i'} = \frac{x}{L}, \\ \int_{0}^{h_{i'+1}} x f_{i'}(x) dx = B_{i'} - B_{i'+1} \end{cases}$$

We derive the following corollary based on the system of equations, which applies to the case where there are two players with $B_1 \geq B_2$ competing on a single battlefield. It is important to note that our corollary aligns with the Nash equilibrium results presented in the paper [19].

Corollary 1. *If there are only two players with $B_1 \geq B_2$ and a single battlefield, then their Nash equilibrium strategies are*

$$F_1(x) = \frac{x}{2B_1}, \quad x \in [0, 2B_1]; \quad F_2(x) = \frac{B_2}{2B_1^2}x + 1 - \frac{B_2}{B_1}, \quad x \in [0, 2B_1].$$

4.2 The Uniqueness of the Nash Equilibrium

Given that there are at least two players with the maximum budget, i.e., the budget vector $\boldsymbol{B} = (B_1, B_2, \cdots, B_n)$ satisfies $B_1 = B_2 \geq B_3 \geq \cdots \geq B_n$, we prove the uniqueness of the Nash equilibrium. With the help of the system of equations, we derive Theorem 3, which establishes the uniqueness of the Nash equilibrium.

Theorem 3. *If the budget vector $\boldsymbol{B} = (B_1, B_2, \cdots, B_n)$ satisfies $B_1 = B_2 \geq B_3 \geq \cdots \geq B_n$, then the Nash equilibrium for the GL game with a single battlefield is unique.*

Assume the budget vector $\boldsymbol{B}$ satisfies $B_1 = B_2 = \cdots = B_{i'} > B_{i'+1} \geq \cdots \geq B_n$. By Theorem 2, we have $0 = h_1 = h_2 = \cdots = h_{i'} < h_{i'+1} \leq \cdots \leq h_n$, and by Lemma 11, $0 = p_1 = p_2 = \cdots = p_{i'} < p_{i'+1} \leq \cdots \leq p_n$. Lemmas 10 and 11 together imply that, in Nash equilibrium, every player's strategy can be expressed in terms of player 2's strategy F_2. Hence it suffices to show that player 2's strategy F_2 is unique. Here, we define $M_i = \prod_{i' \geq i} p_{i'}$ and according to the system of equations above, we can observe (1) in the interval $[0, h_{i'+1}]$, player 2's strategy F_2 depends only on L and $M_{i'+1}$; (2) for $r \in \{i'+1, \cdots, n-1\}$, in $[h_r, h_{r+1}]$ it depends on L and M_{r+1}; (3) in $[h_n, L]$ it depends solely on L. Thus we need only verify the uniqueness of L, $M_{i'+1}$ and M_{r+1}, $r \in \{i'+1, \cdots, n-1\}$. Starting from player n and moving backwards, we can compute each M_i and L uniquely from successive budget differences $B_i - B_{i+1}$. This backward induction shows that L and all relevant M-values are uniquely determined; consequently F_2 is unique, and therefore the Nash equilibrium itself is unique.

5 Analysis of Nash Equilibrium with Multiple Battlefields

In this section, we analyze the properties of the Nash equilibrium in the General Lotto game with multiple battlefields. When there are multiple battlefields, a player may choose to abandon a battlefield by consistently bidding 0 on it. We consider two extreme cases:

- Suppose that all valuations v_{ij} are independently and randomly drawn from some continuous distributions, then with probability one, the condition $\frac{v_{i,j}}{v_{i,j'}} \neq \frac{v_{i',j}}{v_{i',j'}}$ holds for all $i \neq i'$ and $j \neq j'$. We find that the average number of battlefields in which each player participates (i.e., bids larger than 0 with positive probability) becomes arbitrarily close to one as n becomes sufficiently large (see Theorem 5).

- In the symmetric setting, where for any player i, $B_i = B$, and for any two players $i \neq i'$ and any battlefield j, $v_{ij} = v_{i'j} = v_j$, we provide a solution for the Nash equilibrium (see Theorem 6).

Finally, we examine intermediate case between these two extremes by providing an example to illustrate the non-uniqueness of the Nash equilibrium.

Theorem 4. *For any Nash equilibrium and any two battlefields $j \neq j'$, let S and S' represent the sets of players who do not always bid 0 on j and j', respectively. If $\frac{v_{i,j}}{v_{i,j'}} \neq \frac{v_{i',j}}{v_{i',j'}}$ holds for all $i \neq i'$, then the cardinality of the set $S \cap S'$ is not greater than 3.*

Proof. Let $F_{\cdot,\cdot} = (F_{i,j})_{i\in[n],j\in[m]}$ denote an arbitrary equilibrium. By Theorem 2, for any $\delta > 0$, every $i \in S$ should bid in $[L_j - \delta, L_j]$ with positive probability on battlefield j. By Lemma 5, $u_{i,j}(\cdot, F_{-i,j})$ is continuous on $(0, T]$ (the definition of T is given in Lemma 4), then by Lemma 1 we have $u_{i,j}(L_j, F_{-i,j}) = a_i L_j + b_{i,j}$.

Consider an arbitrary $i \in S \cap S'$.

1. If $b_{i,j} = b_{i,j'} = 0$, then $u_{i,j}(L_j, F_{-i,j}) = a_i L_j = v_{i,j}$ and $u_{i,j'}(L_{j'}, F_{-i,j'}) = a_i L_{j'} = v_{i,j'}$, therefore $\frac{v_{i,j}}{v_{i,j'}} = \frac{a_i L_j}{a_i L_{j'}} = \frac{L_j}{L_{j'}}$. The right-hand side does not depend on i. Since $\frac{v_{i,j}}{v_{i,j'}} \neq \frac{v_{i',j}}{v_{i',j'}}$, for any given (j, j') there can be at most one such i.
2. If $b_{i,j} = 0$ and $b_{i,j'} > 0$, by Lemma 2 we have at most one such i given j'.
3. If $b_{i,j} > 0$, again by Lemma 2, we have at most one such i given j.

We conclude that at most three players in the set $S \cap S'$. □

Based on Theorem 4, we can derive the following theorem.

Theorem 5. *Let D_i denote the set of battlefields where player i does not always bid 0, and d_i denote the cardinality of the set D_i. If $\frac{v_{ij}}{v_{ij'}} \neq \frac{v_{i'j}}{v_{i'j'}}$ holds for all pairs of players $i \neq i'$ and all pairs of battlefields $j \neq j'$, we have $\frac{1}{n}\sum_{i=1}^{n} d_i < 1 + m\sqrt{\frac{3}{n}}$.*

Proof. Let S_j denote the set of players who do not always bid 0 on battlefield j, and $S_{j'}$ denote the set of players who do not always bid 0 on j'. By Theorem 4, we have $\sum_{(j,j')} |S_j \cap S_{j'}| \leq 3 \cdot \binom{m}{2}$. Note that $\sum_{(j,j')} |S_j \cap S_{j'}| = \sum_{i=1}^{n} \binom{d_i}{2}$. We establish the following inequality

$$\sum_{i=1}^{n} \binom{d_i}{2} = \sum_{(j,j')} |S_j \cap S_{j'}| \leq 3 \cdot \binom{m}{2} = \frac{3m(m-1)}{2}.$$

Using the Cauchy inequality, we have

$$\sum_{i=1}^{n} \binom{d_i}{2} \geq \frac{1}{2n}(\sum_{i=1}^{n} d_i)^2 - \frac{1}{2}\sum_{i=1}^{n} d_i.$$

Therefore, we obtain

$$\frac{1}{2n}(\sum_{i=1}^{n} d_i)^2 - \frac{1}{2}\sum_{i=1}^{n} d_i \leq \frac{3m(m-1)}{2}.$$

Let $\zeta = \frac{1}{n}\sum_{i=1}^{n} d_i$, we have $n\zeta^2 - n\zeta - 3m(m-1) \leq 0$. Solving this inequality, we obtain $\zeta \leq \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{3m(m-1)}{n}}$. Furthermore,

$$\sqrt{\frac{1}{4} + \frac{3m(m-1)}{n}} < \frac{1}{2} + m\sqrt{\frac{3}{n}}.$$

Finally, we have $\frac{1}{n}\sum_{i=1}^{n} d_i < 1 + m\sqrt{\frac{3}{n}}$. □

Theorem 5 states that for a fixed number of battlefields, the average number of battlefields in which each player participates can be arbitrarily close to one as n becomes sufficiently large. Specifically, for $n = m^2$, the average value is less than 2.73.

Although the computation of the Nash equilibria is highly complex in the multi-player and multi-battlefield setting, it can be computed in certain symmetric settings. The following theorem provides a symmetric solution of the Nash equilibria in a symmetric setting.

Theorem 6. *Suppose we have $B > 0$ such that $B_i = B$ for every i, and for every j we have $v_j > 0$ such that $v_{ij} = v_j$ for every i. Then the strategies given by $F_{ij}(x) = (\frac{vx}{v_j nB})^{\frac{1}{n-1}}$ for every i and j, where $v = \sum_{j\in[m]} v_j$, is a Nash equilibrium.*

Proof. Consider the symmetric Nash equilibrium.

Let e_{ij} denote the expected bid of player i on battlefield j, and L_j denote the upper bound of the support on the battlefield j. Note that all players have the same expected bid on battlefield j, and all players play the same strategy, i.e., $e_{ij} = e_j$ and $F_{ij}(x) = F_j(x)$ for $\forall i$. Therefore, we have

$$u_{ij} = v_j(F_j(x))^{n-1} = \frac{v_j}{L_j}x.$$

So we can derive

$$\begin{cases} F_j(x) = (\frac{x}{L_j})^{\frac{1}{n-1}}, \\ f_j(x) = \frac{1}{n-1}\frac{1}{L_j}(\frac{x}{L_j})^{\frac{2-n}{n-1}}. \end{cases}$$

The expected value of the distribution $F_j(x)$ is equal to the budget. Therefore, we have

$$e_j = \int_0^{L_j} x f_j(x)dx = \int_0^{L_j} x\frac{1}{n-1}\frac{1}{L_j}(\frac{x}{L_j})^{\frac{2-n}{n-1}}dx = \frac{L_j}{n}.$$

Thus we get $L_j = ne_j$. By Lemma 1, we have $\frac{v_1}{L_1} = \frac{v_j}{L_j}$ and $L_j = \frac{v_j}{v_1} L_1$ for $\forall j$. We can obtain $ne_j = \frac{v_j}{v_1} L_1$, thereby $e_j = \frac{v_j L_1}{v_1 n}$.

Due to the budget constraints, we have

$$\sum_{j \in [m]} e_j = \sum_{j \in [m]} \frac{v_j L_1}{v_1 n} = \frac{L_1 v}{v_1 n} = B,$$

we get $L_1 = \frac{v_1 n B}{v}$, and $L_j = \frac{v_j n B}{v}$, $\forall j \in [m]$. Finally, we have $F_j(x) = \left(\frac{x}{\frac{v_j n B}{v}}\right)^{\frac{1}{n-1}} = \left(\frac{vx}{v_j n B}\right)^{\frac{1}{n-1}} = F_{ij}(x)$. □

Following Theorem 6, we derive the following corollary, which applies to the case where there are two players with $B_1 = B_2$ competing across multiple battlefields where the values of these battlefields are symmetric between the players. Our corollary 2 also implies the Nash equilibrium in the symmetric setting established by Roberson and Kovenock [19].

Corollary 2. *If there are only two players with $B_1 = B_2 = B$ and multiple battlefields with $v_{1j} = v_{2j}$ for all j, then*

$$F_{1j}(x) = F_{2j}(x) = \frac{v}{2Bv_j} x, \quad x \in [0, \frac{2Bv_j}{v}],$$

is a Nash equilibrium.

Although we prove in the previous section that Nash equilibrium is unique when at least two players have the maximum budget, this uniqueness does not hold when there are multiple battlefields. Here is an example.

Example. Consider a game with three players and two battlefields. The players have budgets of $B_1 = 10$, $B_2 = 6$, $B_3 = 6$, respectively, and each player assigns a value of 1 to each battlefield.

For this example, we examine two budget vectors:

(1) $\boldsymbol{B}^{(1)} = ((5, 4, 2), (5, 2, 4))$, in which the expectation of bids on the first battlefield is $(5, 4, 2)$ corresponding to the resources invested by player 1, 2, and 3 in that battlefield respectively, and the expected bids on the second battlefield is $(5, 2, 4)$.
(2) $\boldsymbol{B}^{(2)} = ((5, 3, 3), (5, 3, 3))$, in which the expected bids on the first battlefield is $(5, 3, 3)$ and the expected bids on the second battlefield is $(5, 3, 3)$.

Let p_{ij} denote the probability that player i bids 0 on battlefield j, and let L_j denote the upper endpoint of the players' support on battlefield j.

In the first budget vector, it is easy to see that $p_{11} = p_{12}$, $p_{21} = p_{32}$, $p_{31} = p_{22}$ and $L_1 = L_2 = 11.9231$.

In the second budget vector, it follows that $p_{11} = p_{12}$, $p_{21} = p_{22}$, $p_{31} = p_{32}$ and $L_1 = L_2 = 12.5547$. We observe the symmetry in strategy between the two budget vectors $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$.

Therefore, these strategies indeed form a Nash equilibrium.

6 Conclusion

We extend the General Lotto game from a two-player game to a multiplayer game in a general setting, proving the existence of Nash equilibrium in the multiplayer version and providing a detailed characterization of the Nash equilibrium on a single battlefield. Additionally, we establish the uniqueness of Nash equilibrium in certain single-battlefield scenarios. Finally, we generalize the game from a single battlefield to multiple battlefields. Under the setting of multiple battlefields, we find that the average number of battlefields in which each player participates (i.e., bids larger than 0 with positive probability) becomes arbitrarily close to one as n becomes sufficiently large. We also show the non-uniqueness of Nash equilibrium by providing an example.

Acknowledgments. Zihe Wang was supported by the National Natural Science Foundation of China (Grant No. 62172422), the Fund for Building World-Class Universities (Disciplines) of Renmin University of China, and the Key Laboratory of Interdisciplinary Research of Computation and Economics (Shanghai University of Finance and Economics), Ministry of Education. Weiran Shen was supported by the National Natural Science Foundation of China (Grant No. 72192805). Jie Zhang was partially supported by a Leverhulme Trust Research Project Grant (2021–2024) and the EPSRC grant (EP/W014912/1). This work was also supported by the Public Computing Cloud of Renmin University of China.

References

1. Ahmadinejad, A., Dehghani, S., Hajiaghayi, M., Lucier, B., Mahini, H., Seddighin, S.: From duels to battlefields: Computing equilibria of blotto and other games. Mathematics of Operations Research **44**(4), 1304–1325 (2019)
2. Beale, E., Heselden, G.: An approximate method of solving blotto games. Naval Research Logistics Quarterly **9**(2), 65–79 (1962)
3. Behnezhad, S., Dehghani, S., Derakhshan, M., Hajiaghayi, M., Seddighin, S.: Fast and simple solutions of blotto games. Operations Research **71**(2), 506–516 (2023)
4. Boix-Adserà, E., Edelman, B.L., Jayanti, S.: The multiplayer colonel blotto game. In: Proceedings of the 21st ACM Conference on Economics and Computation. pp. 47–48 (2020)
5. Borel, E.: La théorie du jeu et les équations intégrales à noyau symétrique. Comptes rendus de l'Académie des sciences **173**, 1304–1308 (1921)
6. Borel, E.: ... Applications [de la théorie des probabilités] aux jeux de hasard: cours professé à la Faculté des sciences de Paris, vol. 2. Gauthier-Villars (1938)
7. Chandan, R., Paarporn, K., Alizadeh, M., Marden, J.R.: Strategic investments in multi-stage general lotto games. In: 2022 IEEE 61st Conference on Decision and Control (CDC). pp. 4444–4448. IEEE (2022)
8. Chandan, R., Paarporn, K., Kovenock, D., Alizadeh, M., Marden, J.R.: The art of concession in general lotto games. In: International Conference on Game Theory for Networks. pp. 310–327. Springer (2022)
9. Chandan, R., Paarporn, K., Marden, J.R.: When showing your hand pays off: Announcing strategic intentions in colonel blotto games. In: 2020 American Control Conference (ACC). pp. 4632–4637. IEEE (2020)

10. Chowdhury, S.M., Kovenock, D., Sheremeta, R.M.: An experimental investigation of colonel blotto games. Economic Theory **52**, 833–861 (2013)
11. Donahue, K., Kleinberg, J.: Private blotto: Viewpoint competition with polarized agents. arXiv preprint arXiv:2302.14123 (2023)
12. Dziubiński, M.: Non-symmetric discrete general lotto games. International Journal of Game Theory **42**, 801–833 (2013)
13. Hart, S.: Discrete colonel blotto and general lotto games. International Journal of Game Theory **36**(3-4), 441–460 (2008)
14. Hortala-Vallve, R., Llorente-Saguer, A.: Pure strategy nash equilibria in non-zero sum colonel blotto games. International Journal of Game Theory **41**, 331–343 (2012)
15. Iliaev, D., Oren, S., Segev, E.: A tullock-contest-based approach for cyber security investments. Annals of Operations Research **320**(1), 61–84 (2023)
16. Jayanti, S.: Nash equilibria of the multiplayer colonel blotto game on arbitrary measure spaces. arXiv preprint arXiv:2104.11298 (2021)
17. Konrad, K.A., Morath, F.: How to preempt attacks in multi-front conflict with limited resources. European Journal of Operational Research **305**(1), 493–500 (2023)
18. Kovenock, D., Roberson, B.: Coalitional colonel blotto games with application to the economics of alliances. Journal of Public Economic Theory **14**(4), 653–676 (2012)
19. Kovenock, D., Roberson, B.: Generalizations of the general lotto and colonel blotto games. Economic Theory **71**, 997–1032 (2021)
20. Liu, Y., Ni, B., Shen, W., Wang, Z., Zhang, J.: Stackelberg vs. nash in the lottery colonel blotto game. In: Proceedings of the Thirty-Fourth International Joint Conference on Artificial Intelligence, IJCAI-25. pp. 3961–3969. International Joint Conferences on Artificial Intelligence Organization (2025)
21. Liu, Y., Qin, Y., Wang, Z.: Simultaneous all-pay auctions with budget constraints. In: International Computing and Combinatorics Conference. pp. 333–345. Springer (2025)
22. Macdonell, S.T., Mastronardi, N.: Waging simple wars: a complete characterization of two-battlefield blotto equilibria. Economic Theory **58**(1), 183–216 (2015)
23. Myerson, R.B.: Incentives to cultivate favored minorities under alternative electoral systems. American Political Science Review **87**(4), 856–869 (1993)
24. Paarporn, K., Chandan, R., Alizadeh, M., Marden, J.R.: Incomplete and asymmetric information in general lotto games. IEEE Transactions on Automatic Control (2024)
25. Perchet, V., Rigollet, P., Le Gouic, T.: An algorithmic solution to the blotto game using multi-marginal couplings. In: Proceedings of the 23rd ACM Conference on Economics and Computation. pp. 208–209 (2022)
26. Roberson, B.: The colonel blotto game. Economic Theory **29**(1), 1–24 (2006)
27. Sahuguet, N., Persico, N.: Campaign spending regulation in a model of redistributive politics. Economic Theory **28**(1), 95–124 (2006)
28. Shishika, D., Guan, Y., Dorothy, M., Kumar, V.: Dynamic defender-attacker blotto game. In: 2022 American Control Conference (ACC). pp. 4422–4428. IEEE (2022)
29. Vu, D.Q., Loiseau, P.: Colonel blotto games with favoritism: Competitions with pre-allocations and asymmetric effectiveness. In: Proceedings of the 22nd ACM Conference on Economics and Computation. pp. 862–863 (2021)
30. Vu, D.Q., Loiseau, P., Silva, A.: Efficient computation of approximate equilibria in discrete colonel blotto games. In: Proceedings of the Twenty-Seventh International Joint Conference on Artificial Intelligence, IJCAI-18. pp. 519–526. International Joint Conferences on Artificial Intelligence Organization (2018)