Optimal Auction Design with User Coupons in Advertising Systems

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Abstract

Online advertising is a major revenue source for most Internet companies. The advertising opportunities are usually sold to advertisers through auctions that take into account the bids of the advertisers, the click-through rates (CTRs), and the conversion rates (CVRs) of the users. Standard auction design theory perceives both the CTRs and the CVRs as constants. We consider a new auction mechanism that offers coupons to users when displaying the ads. Such coupons allow the user to buy the advertisers' products or services at a lower price, which increases both the CTRs and the CVRs of the ads.

In this paper, we formulate the problem mathematically and perform a systematic analysis. We characterize the set of individually rational and incentive-compatible mechanisms in our setting. Based on the characterization, we identify the optimal strategy of offering coupons that maximizes the platform's expected revenue. We also conduct extensive experiments on both synthetic data and industrial data. Our experiment results show that our mechanism significantly improves both the revenue and welfare of the platform, thereby creating a win-win situation for all parties including the platform, the advertisers, and the user.

1 Introduction

Auction design has become one of the central topics at the intersection of economics and computer science. Various auctions are proposed in the literature over the years [Cramton *et al.*, 2004; Aggarwal *et al.*, 2006; Varian, 2007; Edelman *et al.*, 2007], including the celebrated Vickrey-Clarke-Groves (VCG) auction [Clarke, 1971; Vickrey, 1961; Groves, 1973] and Myerson's optimal auction [Myerson, 1981]. Both auctions are truthful, where bidders are incentivized to submit their true valuations as their bids. The appeal of the VCG auction is this mechanism maximizes social welfare, allocating the item to the bidder with the highest valuation, while Myerson's optimal auction maximizes the seller's revenue for a single-item scenario.

One of the most successful applications of auction theory is online advertising, as online advertising is a major source of revenue for many Internet companies, e.g., Meta, Google and TikTok. According to the statistics by Statista[1], 627 billion U.S. dollars were spent in 2023 on digital advertisements, and the industrial is still rapidly growing.

When applied to online advertising systems, besides the advertisers' bids, the standard auction design theory also needs to consider the relevance scores of the ads. Such relevance scores are usually captured by the click-through rates (CTRs) and the conversion rates (CVRs). Standard auction design theory implicitly assumes that the CTR and CVR are known when the user and the ad are given, and thus treat them as constants during the auction process.

However, in this paper, we consider a setting where the platform can offer coupons to the users when displaying the ads. The users can buy the corresponding advertiser's products or services at a lower price by using the coupons. Thus the coupons can increase both the CTR and the CVR of the ads. In such a setting, we can no longer view the CTRs and CVRs as constants, as offering different coupons can lead to different CTR and CVR improvements. Therefore, we study the problem of how to jointly design the coupon strategy, the allocation rule, and the payment rule to maximize the platform's revenue.

Coupons offer a financial discount off the regular price for users when purchasing a product. In economic studies, different prices may result in different demands, and such a relation is often described by a so-called "demand curve" [Mas-Colell *et al.*, 1995]. The fundamental economic principle known as the law of demand intuitively states that as the price of a good falls, the demand rises [Nicholson and Snyder, 2012]. Therefore, offering coupons to the users increases the buyers' willingness to buy and therefore increases the CTRs and CVRs of the ads. From this point of view, coupons can be

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[1]https://www.statista.com/topics/990/global-advertising-market/#topicOverview

viewed as a way to balance the price and demand for advertisers. Most advertisers lack the ability to draw the entire demand curve for their products, while the platform can predict the CTR and CVR changes accurately with its massive amount of data. By leveraging such additional knowledge, the platform can design pricing strategies to better serve the interests of the advertisers.

In our paper, we analyze the auction design problem with user coupons theoretically and conduct experiments to evaluate its performance. It is worth mentioning that, the first online ad auction mechanism that consider user coupons is proposed by Ni *et al.* [2023]. They empirically show that such mechanisms indeed improve the revenue of the platform. However, they consider the auto-bidding setting and their mechanism is not incentive compatible when the advertisers have quasi-linear utilities.

We summarize our contributions in this paper as follows:

- We model the auction design problem with user coupons mathematically and characterize the set of incentive compatible and individually rational mechanisms.
- Based on the above result, we formulate the revenue-maximization problem as a mathematical program, and provide an optimal mechanism.
- We conduct extensive experiments with both synthetic and industrial data sets to demonstrate the performance of our mechanism.

1.1 Related Work

Our paper is clearly related to the optimal mechanism design [Myerson, 1981; Maskin *et al.*, 1981; Pavlov, 2011; Wang and Tang, 2014]. Another line of work aims to improve revenue for commonly used mechanisms in practice, such as the second-price auction, the generalized second-price auction, and the VCG auction. These methods include optimizing reserve prices [Hartline and Roughgarden, 2009; Balseiro *et al.*, 2021], boosts [Golrezaei *et al.*, 2021; Deng *et al.*, 2021], offering coupon to advertisers [Shen *et al.*, 2020; Shen *et al.*, 2021].

Recently, Ni *et al.* [2023] proposed a new mechanism that offers coupons to users to increase the platform's revenue. However, they study the problem in an auto-bidding setting, and in their model, the cost of offering coupons is covered by the advertisers, making their mechanism nontruthful for advertisers with quasi-linear utilities.

Another relevant topic is the effect of offering coupons on consumer behaviors [Gopalakrishnan and Park, 2021; Ali and Muhammad, 2021; Li *et al.*, 2020; Albert and Goldenberg, 2022]. They show that coupons in general have positive effects on consumers' purchasing behavior, and therefore increase sales volumes.

2 Preliminaries

In this section, we describe how we mathematically model the problem of designing auction mechanisms with user coupons and introduce some basic definitions into our setting.

2.1 Model

An online advertising platform (e.g., a search engine or a short-form video platform) sells an ad slot to n advertisers, denoted by $[n] := \{1, 2, \cdots, n\}$. Each advertiser $i \in [n]$ has a valuation v_i, which is drawn from a cumulative distribution function $F_i(v_i)$. We assume that $F_i(v_i)$ is differentiable with density function $f_i(v_i)$ and that $f_i(v_i)$ is bounded and supported on $\mathcal{V} = [0, \overline{v}]$, i.e., $0 < f_i(v_i) < \infty, \forall v_i \in \mathcal{V}$. The valuation can be interpreted as the advertiser's willingness to pay for one click. When a user enters a query into a search engine or scrolls down on a short-form video app, an auction is triggered. In the standard auction setting, the advertisers first submit their bids to the platform. Denote the bid of advertiser i by $b_i \in \mathcal{B}$, which may be different from the true valuation v_i. The platform determines who wins the auctions and how much the winner should pay, and then shows the winner's ad to the user. During the auction process, the platform usually also takes into account the click-through rate (CTR) and the conversion rate (CVR) of each ad, where the CTR is the probability of the user clicking on an ad and the CVR is the probability of buying the advertiser's product conditioned on clicking the ad.

In this paper, we consider a setting where the platform also offers a coupon to the user when displaying the ad. Though offered by the platform, the coupon can be used when the user buys the advertiser's products. Such a coupon can clearly make the ad and the advertiser's product more attractive to the user and thus increase both the CTR and the CVR of the ad.

Therefore, different from the standard auction setting, besides the winner and the payment, the platform also needs to determine a coupon $c_i \in \mathcal{C} = [0, \overline{c}]$, for each advertiser $i \in [n]$. We use the function $h_i := \mathcal{C} \mapsto (0, 1]$ to represent the CTR of advertiser i's ad when coupon c_i is offered to the user. In particular, $h_i(0)$ represents the CTR without a coupon. An ad is usually more attractive to a user if a larger coupon is offered. Thus we assume $h_i(c_i)$ is a strictly increasing function. Similarly, we use $r_i := \mathcal{C} \mapsto [0, 1]$ to represent the CVR of advertiser i's ad when coupon c_i is offered, which is also assumed to be strictly increasing. For simplicity, throughout the paper, we also make the assumption that both $h_i(c_i)$ and $r_i(c_i)$ are differentiable. If the user buys the advertiser's product, the payment of the user is the price of the product less the coupon c_i, and the platform also pays c_i to the advertiser (directly or indirectly through the user) to ensure that the total payment is still the price of the product. Therefore, the expected cost of the coupon can be written as $h_i(c_i)r_i(c_i)c_i$.

In our setting, apart from an allocation rule and a payment rule, a mechanism should also include a coupon function. In this paper, we allow the coupon c_i to depend on the bid b_i submitted by advertiser i. Formally, we have the following definition.

Definition 1 (Mechanism)**.** *A mechanism $\mathcal{M}$ is a tuple (ϕ, x, p), where:*

- $\phi = (\phi_1, \phi_2 \cdots, \phi_n)$ *is the coupon function, where* $\phi_i : \mathcal{B} \mapsto \mathcal{C}$ *maps the bid* b_i *of advertiser* i *to the coupon* c_i *to be offered to the user;*
- $x = (x_1, x_2, \cdots, x_n)$ *is the allocation rule, where* x_i :

$\mathcal{B}^n \mapsto [0,1]$ *takes the bid profile as input and outputs the probability of advertiser* i *winning the auction;*

- $p = (p_1, p_2, \cdots, p_n)$ *is the payment function, where* $p_i : \mathcal{B}^n \mapsto \mathbb{R}_+$ *takes the bid profile as input and outputs the payment of advertiser* i *to the platform.*

Let $\mathbf{v} = (v_1, \cdots, v_n)$ be the valuation profile of all advertisers and $\mathbf{v}_{-i} = (v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n)$ the valuation profile of all advertisers except advertiser i. We assume that the valuation of the n advertisers are independent random variables. Thus, the joint density function $f := \mathcal{V}^n \mapsto \mathbb{R}_+$ for the vector $\mathbf{v} = (v_1, \cdots, v_n)$ of individual valuations is

$$f(\mathbf{v}) = \prod_{i \in [n]} f_i(v_i).$$

Also, we define $f_{-i} := \mathcal{V}^{n-1} \mapsto \mathbb{R}_+$ as the join density function of vector $\mathbf{v}_{-i}$,

$$f_{-i}(\mathbf{v}_{-i}) = \prod_{j \in [n] \setminus i} f_j(v_j).$$

Similarly, let $\mathbf{b} = (b_1, \cdots, b_n)$ and $\mathbf{b}_{-i} = (b_1, \cdots, b_{i-1}, b_{i+1}, \cdots, b_n)$ denote all advertisers' bids profile and all advertisers' bids profile except advertiser i.

The auction procedure is as follows:

1. All advertisers simultaneously report their bid b_i to the platform;
2. After receiving the bids, the platform calculates the coupon c_i for each advertiser $i \in [n]$ according to the coupon function $\phi_i(b_i)$;
3. The platform computes the CTR h_i and the CVR r_i of each ad i based on the coupon c_i;
4. The platform determines the winner and the payment according to the allocation rule x and the payment rule p;
5. The platform displays the ad and offers the coupon together to the user.

In this paper, we consider cost-per-click auctions, i.e., an advertiser only pays the platform when their ad is clicked by the user. Suppose all advertisers have quasi-linear utilities. Formally, if the bid profile is $\mathbf{b}$, in mechanism $\mathcal{M}$, the expected utility of advertiser i is

$$u_i(\mathcal{M}, \mathbf{b}) = x_i(b_i, \mathbf{b}_{-i}) h_i(\phi_i(b_i)) v_i - p_i(b_i, \mathbf{b}_{-i}).$$

Since the platform needs to provide coupons to the users, the revenue of the platform can be represented as the payments from the advertisers in the auction minus the cost of providing the coupon:

$$\textbf{Rev}(\mathbf{b}) = \sum_{i=1}^{n} p_i(\mathbf{b}) - \sum_{i=1}^{n} x_i(\mathbf{b}) h_i(\phi_i(b_i)) r_i(\phi_i(b_i)) \phi_i(b_i).$$

Naturally, a mechanism should satisfy the following constraint as the platform only has a single slot for sale.

$$\sum_{i=1}^{n} x_i(\mathbf{b}) \leq 1, \forall \mathbf{b}. \tag{1}$$

2.2 Direct Feasible Auction Mechanisms

A mechanism induces a Bayesian game among the advertisers. In such a game, an ad advertiser may have incentives to place a bid different from their true valuation if the bid gives them a better utility. There is a special class of mechanisms where the advertisers' utilities are maximized by submitting their true valuations. Such mechanisms are called *truthful* mechanisms. Before formally defining truthfulness, we first introduce some useful notations. Let $U_i(\mathcal{M}, v_i)$ be the expected utility of advertiser i with valuation v_i, i.e.,

$$\begin{aligned} & U_i(\mathcal{M}, v_i) \\ = & \int_{-i} [x_i(v_i, \mathbf{v}_{-i}) h_i(\phi_i(v_i)) v_i - p_i(v_i, \mathbf{v}_{-i})] f_{-i}(\mathbf{v}_{-i}) \, \mathrm{d}\mathbf{v}_{-i}. \end{aligned}$$

For ease of presentation, we also define

$$X_i(v_i) = \int_{-i} x_i(v_i, \mathbf{v}_{-i}) f_{-i}(\mathbf{v}_{-i}) \, \mathrm{d}\mathbf{v}_{-i},$$
$$P_i(v_i) = \int_{-i} p_i(v_i, \mathbf{v}_{-i}) f_{-i}(\mathbf{v}_{-i}) \, \mathrm{d}\mathbf{v}_{-i}.$$

to be the interim allocation rule and the interim payment rule, which are the expected winning probability and the expected payment of advertiser i when the valuation is v_i.

Definition 2 (Incentive Compatibility (IC))**.** *A mechanism is incentive compatible or truthful, if all advertisers reporting* $b_i = v_i$ *form a Bayes-Nash equilibrium, i.e.,*

$$U_i(\mathcal{M}, v_i) \geq U_i(\mathcal{M}, s), \forall s \in \mathcal{V}, \forall i \in [n]. \tag{2}$$

According to the celebrated *revelation principle* [Myerson, 1979; Gibbard, 1973], we can, without loss of generality, restrict our attention to truthful mechanisms.

Another important property is *individual rationality*, which ensures the participation of the advertisers.

Definition 3 (Individual Rationality (IR))**.** *A mechanism is individually rational if participating in the auction always leads to a non-negative expected utility for all advertisers, i.e., for advertiser* i *with valuation* v_i*, we have*

$$U_i(\mathcal{M}, v_i) \geq 0. \tag{3}$$

In this paper, we consider the set of *feasible* mechanisms. Formally,

Definition 4. *A mechanism* $\mathcal{M} = (\phi, x, p)$ *is feasible if it does not oversell, and satisfies both the IC and IR properties, i.e.,* $\mathcal{M}$ *satisfies Equation* (1), (2) *and* (3).

The following lemma characterizes feasible mechanisms.

Lemma 1. *A mechanism* (ϕ, x, p) *is feasible, if and only if for any valuation profile* $\mathbf{v}$*, the following conditions hold:*

1. *Function* $h_i(\phi_i(v_i)) X_i(v_i)$ *is monotone non-decreasing with respect to* v_i*.*
2. *The expected payment of advertiser* i *satisfies*

$$P_i(v_i) = v_i h_i(\phi_i(v_i)) X_i(v_i) - \int_0^{v_i} h_i(\phi_i(s)) X_i(s) \, \mathrm{d}s. \tag{4}$$

3. *The allocation rule satisfies*

$$\sum_{i=1}^{n} x_i(\mathbf{v}) \leq 1, \forall \mathbf{v}. \tag{5}$$

According to Equation (4) in Lemma 1, to ensure incentive compatibility, the advertiser's payment should take into consideration the CTR's increase resulted from the coupons. However, it may be difficult to jointly design the coupon function $\phi_i(v_i)$ and the allocation rule $X_i(v_i)$ to ensure that the function $h_i(\phi_i(v_i))X_i(v_i)$ is monotone non-decreasing, especially when the CTR function $h_i(\cdot)$ is represented as a neural network model in most industrial applications. The following corollary gives a sufficient condition in which the coupon function and the allocation rule can be decoupled.

Corollary 1. *The mechanism $\mathcal{M} = (\phi, x, p)$ is feasible if the following conditions hold:*

1. *Function $\phi_i(v_i)$ and $X_i(v_i)$ are both monotone non-decreasing with respect to v_i;*
2. *Equation* (4) *and* (5) *are satisfied.*

In the standard auction design setting with no coupons, it is well-known that the monotonicity of $X_i(v_i)$ together already implies the incentive compatibility property. The above corollary shows that any such mechanism is also incentive-compatible in our setting, as long as the coupon function is monotone. Therefore, applying any monotone coupon function to an existing truthful mechanism does not violate the IC property, making such coupon designs easier to implement.

2.3 Revenue Maximization Mechanism

In this section, we characterize revenue maximization mechanisms in our setting. Before stating our results, we define a different version of the virtual value function here.

Definition 5. *For each advertiser i, define*

$$J_i(v_i) = \left[v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} - r_i(\phi_i(v_i))\phi_i(v_i)\right] h_i(\phi_i(v_i)).$$

The standard definition of the virtual value function is:

$$\kappa_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}.$$

Our definition is different in that we also include the coupon function. The standard definition of the virtual value function is irrelevant to the mechanism, while here we need to take into consideration the coupon function, which is part of the the mechanism. Using the standard definition, our virtual value function can be written as:

$$J_i(v_i) = \left[\kappa_i(v_i) - r_i(\phi_i(v_i))\phi_i(v_i)\right] h_i(\phi_i(v_i)).$$

The following Lemma 2 is similar to the Myerson lemma [Myerson, 1981] and is helpful for designing optimal auctions.

Lemma 2. *The revenue of any feasible mechanism $\mathcal{M} = (\phi, x, p)$ can be written as:*

$$\int_{\mathbf{v}} \left[\sum_{i=1}^{n} J_i(v_i) x_i(\mathbf{v})\right] f(\mathbf{v})\,\mathrm{d}\mathbf{v}. \tag{6}$$

3 The Optimal Mechanism

According to Lemma 2, to maximize revenue, the platform should design the coupon function $\phi_i(v_i)$ and the allocation rule $x_i(\mathbf{v})$ to maximize Equation (6). We choose to design the two functions separately in order to make the task of optimizing Equation (6) easier, and then show that although separately designed, the two functions together form a feasible mechanism.

We first design the coupon function $\phi_i(v_i)$ to maximize each $J_i(v_i)$. For ease of presentation, we define the function

$$\begin{aligned} \varphi_i(c_i, v_i) &= \left[v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} - r_i(c_i)c_i\right] h_i(c_i) \\ &= \left[\kappa_i(v_i) - r_i(c_i)c_i\right] h_i(c_i). \end{aligned} \tag{7}$$

Clearly, by plugging in any coupon function $c_i = \phi_i(v_i)$, we get $\varphi_i(\phi_i(v_i), v_i) = J_i(v_i)$.

View v_i as a parameter and let correspondence $\Phi_i(v_i)$ be the set of maximizers of function $\varphi_i(c_i, v_i)$.

$$\Phi_i(v_i) = \left\{c_i \in \mathcal{C} \mid \varphi_i(c_i, v_i) \geq \varphi_i(c_i', v_i), \forall c_i' \in \mathcal{C}\right\}. \tag{8}$$

For any v_i, function $\varphi_i(c_i, v_i)$ is a clearly continuous and bounded function of c_i, thus $\Phi_i(v_i)$ is always a non-empty set according to the extreme value theorem. Now we define a special coupon function[2] as

$$\phi_i(v_i) = \max_{c_i \in \Phi_i(v_i)} c_i. \tag{9}$$

It is clear that such a coupon function maximizes $\varphi_i(c_i, v_i)$ in a point-wise way, i.e., for any v_i, $\phi_i(v_i)$ optimizes $\varphi_i(c_i, v_i)$. Thus the virtual value function $J_i(v_i)$ is also maximized with such a choice:

$$J_i(v_i) = \varphi_i(\phi_i(v_i), v_i) = \max_{c_i \in \mathcal{C}} \varphi_i(c_i, v_i). \tag{10}$$

Before we discuss our optimal mechanism, we first introduce the increasing difference property of a function and present some related results that will be useful later.

Definition 6 (Increasing Difference)**.** *A function $g : \mathcal{A} \times \mathcal{B} \mapsto \mathbb{R}$ has increasing differences in (a, b), if for all $a' \geq a$ and $b' \geq b$,*

$$g(a', b') - g(a, b') \geq g(a', b) - g(a, b).$$

If we view b as a parameter of function g, then the difference of the function values at a' and a is a function of b. Increasing difference means the difference function is an increasing function of the parameter b. The increasing difference also implies the difference function is an increasing function of the parameter a.

Note that the property of increasing difference is ubiquitous in game theory. It is applied to cutoff strategies in Bayesian games [Akerlof, 1970] and more recently in supermodular games [Levin, 2003].

The following lemma gives a sufficient condition for the virtual value function $J_i(v_i)$ to be non-decreasing.

[2] A more careful definition would be $\phi_i(v_i) = \sup_{c_i \in \Phi_i(v_i)} c_i$. However, according to Berge's maximum theorem [Berge, 1963; Ok, 2007], The set $\Phi_i(v_i)$ is non-empty and compact. Therefore, $\sup_{c_i \in \Phi_i(v_i)} c_i$ is attainable in $\mathcal{C}$ and $\sup$ can be replaced by $\max$.

Lemma 3. *If function $\varphi_i(c_i, v_i)$ has increasing differences in (c_i, v_i), then $J_i(v_i)$ is monotone non-decreasing with respect to v_i.*

Now we introduce a technical condition called regularity [Myerson, 1981], and discuss two cases based on whether a problem instance is regular or not.

Definition 7 (Regularity). *A problem instance is regular, if for all advertisers, function $\kappa_i(v_i)$ is monotone non-decreasing.*

The above regularity condition is a standard definition in the literature. Here, we identify an equivalent condition.

Lemma 4. *A problem instance is regular, if and only if function $\varphi_i(c_i, v_i)$ has increasing differences for all i.*

3.1 The Regular Case

Now we design the allocation rule $x_i(\mathbf{v})$ in the regular case. It is clear that to maximize Equation (6), the platform should allocate the slot to the advertiser with the highest $J_i(v_i)$ among those with $J_i(v_i) \geq 0$. We show that this design, together with the choice of $\phi_i(v_i)$, forms a feasible mechanism. Let $M(\mathbf{v}) = \{k \mid J_k(v_k) \geq J_{k'}(v_{k'}), \forall k' \in [n]\}$ be the set of advertisers with the highest virtual values. We present our optimal mechanism for the regular case in theorem 1.

Theorem 1. *If a problem instance is regular, then the following mechanism $\mathcal{M} = (\phi, x, p)$ is an optimal mechanism:*

$$\phi_i(v_i) = \max\{\Phi_i(v_i)\},$$

$$x_i(\mathbf{v}) = \begin{cases} 1 & \text{if } i = \min\{M(\mathbf{v})\} \text{ and } J_i(v_i) \geq 0 \\ 0 & \text{otherwise} \end{cases}, \tag{11}$$

$$p_i(\mathbf{v}) = h_i(\phi_i(v_i)) x_i(v_i, \mathbf{v}_{-i}) v_i - \int_0^{v_i} h_i(\phi_i(s)) x_i(s, \mathbf{v}_{-i}) \,\mathrm{d}s, \tag{12}$$

where $\Phi_i(v_i)$ is defined in Equation (8)*.*

Proof. By setting $\phi_i(v_i) = \max\{\Phi_i(v_i)\}$, the function $\varphi_i(c_i, v_i)$ is maximized pointwisely. Thus, the function $J_i(v_i)$ is maximized. And by setting $x_i(\mathbf{v}) = 1$ for the advertiser i with the highest and non-negative $J_i(v_i)$, the platform's expected revenue (Equation (6)) is maximized. Therefore, the mechanism defined in Theorem 1 is revenue-optimal.

Now we show that it is also feasible. In particular, we show that it satisfies the conditions in Corollary 1. According to Topkis's Theorem [Topkis, 1978], $\phi_i(v_i)$ is a monotone non-decreasing function. Since the problem instance is regular, we have that $J_i(v_i)$ is monotone non-decreasing. It follow that, given any $\mathbf{v}_{-i}$, the allocation rule $x_i(v_i, \mathbf{v}_{-i})$ is a monotone non-decreasing function of v_i. Taking expectation over $\mathbf{v}_{-i}$, we have that $X_i(v_i)$ is also monotone non-decreasing. Therefore, the first condition of Corollary 1.

As for the second condition, notice that Equation (5) clearly holds, and Equation (4) immediately follows from taking the expectation over $\mathbf{v}_{-i}$ on both sides of Equation (12). $\square$

3.2 The Irregular Case

Without the regularity condition, the function $J_i(v_i)$ is not a monotone non-decreasing function. This means the allocation rule defined in Equation (11) is also not monotone non-decreasing. As a result, the expected allocation $X_i(v_i)$ may also not be monotone, leading to possible violations of the first condition in Lemma 1. To address this issue, we apply the so-called *ironing* trick to the function $\kappa_i(v_i)$ to ensure that the ironed function $\overline{\kappa}_i(v_i)$ is monotone. Therefore, the corresponding function $\overline{\varphi}_i(c_i, v_i)$ has increasing differences. We show that replacing $\varphi_i(c_i, v_i)$ by $\overline{\varphi}_i(c_i, v_i)$ yields a feasible and optimal mechanism. The following definition of ironing is a standard technique in the literature.

Definition 8 (Ironing [Myerson, 1981]). *Let $q_i = F_i(v_i) \in [0, 1]$ and define $k_i(q_i) = \kappa_i\left(F_i^{-1}(q_i)\right)$. The ironing procedure is as follows.*

1. *Let $K_i(q_i)$ be the integral of $k_i(q_i)$:*
$$K_i(q_i) = \int_0^{q_i} k_i(s) \,\mathrm{d}s;$$
2. *Define $L_i : [0, 1] \mapsto \mathbb{R}$ be the "convex hull" of the graph of function $K_i(q_i)$:*
$$L_i(q_i) = \min_{q^1, q^2, \gamma} \left\{\gamma K_i(q^1) + (1-\gamma) K_i(q^2)\right\},$$
where $q^1, q^2, \gamma \in [0, 1]$ and $\gamma q^1 + (1-\gamma) q^2 = q_i$;
3. *Let $\ell_i(q_i)$ be the derivative of $L_i(q_i)$:*
$$\ell_i(q_i) = L_i^{'}(q_i);$$
4. *Obtain $\overline{\kappa}_i(v_i)$ by variable substitution:*
$$\overline{\kappa}_i(v_i) = \ell_i(q_i) = \ell_i(F_i(v_i)).$$

Based on the ironed $\overline{\kappa}_i(v_i)$, we can define:

$$\overline{\varphi}_i(c_i, v_i) = h_i(c_i)[\overline{\kappa}_i(v_i) - r_i(c_i) c_i]$$
$$\overline{\Phi}_i(v_i) = \{c_i \in \mathcal{C} \mid \overline{\varphi}_i(c_i, v_i) \geq \overline{\varphi}_i(c_i', v_i), \forall c_i' \in \mathcal{C}\},$$
$$\overline{J}_i(v_i) = \overline{\varphi}_i(\overline{\phi}_i(v_i), v_i).$$

After the ironing procedure, the function $\overline{\kappa}_i(v_i)$ becomes a monotone non-decreasing function. The monotonicity of both $h_i(c_i)$ and $\overline{\kappa}_i(v_i)$ implies the ironed function $\overline{\varphi}_i(c_i, v_i)$ has increasing difference.

Before we present our main result, we prove a lemma that is helpful when deriving the optimal mechanism in the irregular case.

Lemma 5. *For any feasible mechanism (ϕ, x, p), the platform's revenue can be written as*

$$\int_{\mathbf{v}} \left[\sum_{i=1}^n \overline{\varphi}_i(\phi_i(v_i), v_i) x_i(\mathbf{v})\right] f(\mathbf{v}) \,\mathrm{d}\mathbf{v} + \int_{\mathbf{v}} \left[\sum_{i=1}^n \left[k_i(F_i(v_i)) - \ell_i(F_i(v_i))\right] h_i(\phi_i(v_i)) x_i(\mathbf{v})\right] f(\mathbf{v}) \,\mathrm{d}\mathbf{v}, \tag{13}$$

Directly optimizing Equation (6) may result in an infeasible mechanism. Lemme 5 states that we can optimize Equation (13) instead.

Theorem 2. *In an irregular problem instance, define*

$$\overline{M}(\mathbf{v}) = \{k \mid \overline{J}_k(v_k) \geq \overline{J}_{k'}(v_{k'}), \forall k' \in [n]\}.$$

Then the following mechanism $\overline{\mathcal{M}} = (\overline{\phi}, \overline{x}, \overline{p})$ *is an optimal mechanism:*

$$\overline{\phi}_i(v_i) = \max\{\overline{\Phi}_i(v_i)\},$$

$$\overline{x}_i(\mathbf{v}) = \begin{cases} 1 & \text{if } i = \min\{\overline{M}(\mathbf{v})\} \text{ and } \overline{J}_i(v_i) \geq 0 \\ 0 & \text{otherwise} \end{cases}, \tag{14}$$

$$\overline{p}_i(\mathbf{v}) = h_i(\overline{\phi}_i(v_i))\overline{x}_i(v_i, \mathbf{v}_{-i})v_i - \int_0^{v_i} h_i(\overline{\phi}_i(s))\overline{x}_i(s, \mathbf{v}_{-i})\, \mathrm{d}s, \tag{15}$$

Note that for the regular case, Theorem 2 is the same as Theorem 1. Theorem 2 can be applied to any general case.

4 Properties of the Optimal Mechanism

In this section, we derive properties of the optimal mechanism, and try to understand the benefits brought by including user coupons in our setting. We first identify a simple case where the platform should not offer a coupon.

Lemma 6. *The optimal mechanism does not offer coupons for advertiser i if $\overline{\kappa}_i(v_i) \leq 0$.*

The following result shows that if the optimal mechanism offers a coupon, then the marginal cost of the coupon should equal to the marginal gain in revenue.

Lemma 7. *If v_i satisfies $\bar{\phi}_i(v_i) > 0$, then we have:*

$$\bar{\kappa}_i(v_i) \left.\frac{\mathrm{d}}{\mathrm{d}c_i} h_i(c_i)\right|_{c_i = \bar{\phi}_i(v_i)} = \left.\frac{\mathrm{d}}{\mathrm{d}c_i}[h_i(c_i)r_i(c_i)c_i]\right|_{c_i = \bar{\phi}_i(v_i)}.$$

If advertiser i's ad is displayed, the probability that the user buys the advertiser's product is $h_i(c_i)r_i(c_i)$. So $h_i(c_i)r_i(c_i)c_i$ is the expected cost of the platform due to providing the coupon. In the standard auction setting, when bidding v_i, $\overline{\kappa}_i(v_i)$ can be regarded as the contribution of advertiser i to the platform's revenue for one click, and the per-impression contribution is then $\overline{\kappa}_i(v_i)h_i(0)$. However, in our setting, the platform's revenue also comes from the CTR increase for offering coupons to the users, i.e., $\overline{\kappa}_i(v_i)h_i(c_i)$. Lemma 7 shows that the optimal coupon should be such that the marginal coupon cost is exactly covered by the marginal revenue gain.

5 Experiments

We tested our proposed mechanism by conducting a series of experiments, using both synthetic and industrial data. We compared it with three baseline methods: Myerson Auction, Auction with Constant Coupon Value (CCV), and Auction with Randomized Coupon Value (RCV). The Myerson Auction doesn't offer coupons, while CCV and RCV offer coupons to users. The coupon values in CCV are constant, while in RCV they're randomized. The key difference between CCV, RCV, and our method is the coupon function. In our method, coupon values depend on the advertisers' bids, unlike in CCV and RCV. However, all three methods meet the conditions of Lemma 1, so the optimal bidding strategy is to bid truthfully.

5.1 Synthetic Data

We generate a data set using a method similar to the one in [Ni *et al.*, 2023]. We consider a pre-defined coupon set that contains only several possible coupon values, and we generate the corresponding CTRs and CVRs. We suppose there are $n = 8$ advertisers and that they participate in $m = 100,000$ auctions. In each auction, each advertiser's values are drawn from the same log-normal distribution ($\ln v_i \sim N(3, 1)$). For each advertiser i, the CTR without coupon $h_i(0)$ is drawn from the uniform distribution $U(0.005, 0.5)$, and the CVR without coupon $r_i(0)$ is also drawn from the same uniform distribution. The coupon set contains only 4 different coupons: $\mathcal{C} = \{0, 2, 4, 8\}$. For each advertiser i, we let $h_i(2) = 1.1 \cdot h_i(0), h_i(4) = 1.2 \cdot h_i(0), h_i(8) = 1.3 \cdot h_i(0)$, and $r_i(2) = 1.1 \cdot r_i(0), r_i(4) = 1.2 \cdot r_i(0), r_i(8) = 1.3 \cdot r_i(0)$. We use $80,000$ auctions to select the best coupon strategies for the second and third baseline algorithms. We use the other $20,000$ auctions to verify the experimental results between our method and the baseline methods.

Method

We simulate each auction via the following procedures in our methods.

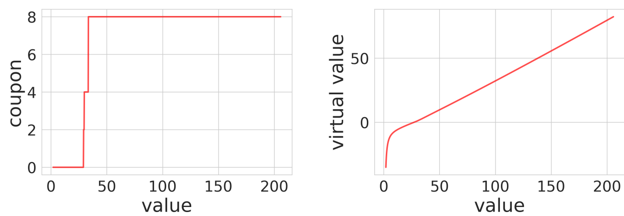
1. For each advertiser i, we enumerate all coupons and compute each advertiser's best coupon using our mechanism.
2. We rank advertisers according to the virtual values defined in Definition 5.
3. The advertiser with the highest virtual value wins the auction and his payment is computed via payment rule (Equation (12)) in Theorem 1.

Results

We compare the performance of our mechanism with the other three baseline auctions mentioned above. [3]

We first verify that in this experiment environment, both $\phi_i(v_i)$ and $J_i(v_i)$ are monotone non-decreasing with respect to v_i for each i. Since all advertisers' values, CTRs, and CVRs are drawn from i.i.d. distributions, we only need to verify the monotonicity of a single advertiser. We sample one CTR and one CVR for one advertiser and use different valuations to compute the corresponding coupon and virtual value according to the coupon function in Theorem 1 and virtual value function defined in Definiton 5 for each valuation. The results are shown in Figure 1(a) and 1(b). The monotonicity of both $\phi_i(v_i)$ and $J_i(v_i)$ indicates that we do not need to apply the ironing technique to guarantee the IC condition.

[3] We choose not to compare our mechanism against [Ni *et al.*, 2023] since their mechanism is not truthful in our setting, and it is not clear how the advertisers would bid under their mechanism.



(a) Coupon function (b) Virtual value function

Figure 1: The advertiser's coupon and virtual value as a function of the value.

Then we simulate the 20,000 auctions and compute the average revenue, social welfare, and the advertisers' total utilities. The results are shown in Table 1. Compared to the Myerson auction without offering coupons, the auction with constant coupon values, and the auction with randomized coupon values, our mechanism increases revenue by about 19.76%, 1.9%, and 6.4% respectively. Also, our mechanism increases the advertisers' total utility by 32.58%, 10.47% and 15.52% respectively, and the social welfare by 26%, 6.2%, and 10.96% respectively. According to the synthetic data-generating method, the advertisers and the users are all symmetric. Thus, the constant coupon value strategy is better than the randomized coupon value strategy. These results show that our mechanism leads to a win-win situation by increasing social welfare and revenue simultaneously. Our mechanism is also clearly beneficial to the users as they can buy the products or services provided by the advertisers with a lower price.

Method	Revenue	Social Welfare	Utility
Myerson	14.32	28.41	14.09
CCV	16.83	33.74	16.91
RCV	16.12	32.29	16.17
Ours	**17.15**	**35.83**	**18.68**

Table 1: Synthetic data results

5.2 Industrial Data

We also conduct experiments using an industrial data set. The industrial data we used comes from Kuaishou, a major short-form video and live-streaming platform. It is worth mentioning that the platform charges the advertisers in a per-impression fashion. Thus the CTR value is unimportant. In this case, we re-define the CVR $r_i(\cdot)$ to be the probability of the user buying a product or service from the advertiser, conditioned on seeing the ad. We also need to re-define the virtual value function as follows:

$$\hat{J}_i(v_i) = r_i(\phi_i(v_i))\big[\kappa_i(v_i) - \phi_i(v_i)\big] \tag{16}$$

Our mechanism can also be easily modified accordingly, thus we omit the detailed definition here.

We select 1,000 ads to conduct the experiment. In this experiment, we trained the CVR prediction model instead of randomly sampling them from certain distributions. To collect enough data for the training, we offer different coupons for these ads in 14 days and sampled 300,000 data records. Our data contains numerous features for users, products, and advertisements, respectively.

We sample 100 advertisers and 100,000 users to conduct the auction process. Then, we randomly choose 80,000 users to select the best coupon strategies for the second and third baseline algorithms. We use the rest to verify the experimental results between our method and the baseline methods.

Method

We conduct the experiment using the following procedure:

1. We train a CVR model using the collected data. The input of the model consists of all the features mentioned above, as well as the coupon value in order to predict the CVR for different coupons.
2. For each auction, we sample a user from the set of users and predict each advertiser's CVR using our CVR model. Then we discrete the coupon space for each advertiser, compute the $\phi_i(c_i, v_i)$ for each possible coupon c_i, and choose the maximum one as the optimal coupon.
3. We rank advertisers according to the modified virtual value function (Equation (16)) and allocate the slot to the advertiser with the highest virtual value and compute his payment via the payment rule (Equation (12)).

Results

The results are shown in Table 2, where we normalize the results using the advertiser's utility in Myerson Auction. Compared with the other three baseline methods, our mechanism increases revenue by 28%, 27%, and 16% respectively. Our mechanism increases the advertisers' total utility by 313%, 91%, and 43% respectively. Our mechanism increases social welfare by 67%, 43%, and 24% respectively.

Method	Revenue	Social Welfare	Utility
Myerson	6.33	7.33	1.00
CCV	6.42	8.58	2.16
RCV	6.98	9.87	2.89
Ours	**8.13**	**12.26**	**4.13**

Table 2: Industrial data results

6 Conclusion

In this paper, we consider the auction design problem of online ad auction with user coupons. Our mechanism space contains that of the standard auction setting as a special subset. Thus, the optimal mechanism in our setting can achieve a higher revenue than the celebrated Myerson auction. We characterize the set of IR and IC mechanisms and also provide the optimal mechanism in closed form. Such a mechanism can be viewed as optimizing the product price for the advertisers to balance price and demand by using additional knowledge possessed by the platform. Our experiments show that our mechanism leads to a win-win situation for the platform, the advertisers, and the users.

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Appendix

A Omitted Proofs in Section 2

A.1 Proof of Lemma 1

Proof. We first show that a feasible mechanism must satisfy the 3 conditions. We only prove the first two conditions as the third one is obvious.

By definition, a feasible mechanism is incentive-compatible. Therefore, for any s,

$$\begin{aligned} &h_i(\phi_i(v_i))X_i(v_i)v_i - P_i(v_i) \\ \geq &h_i(\phi_i(s))X_i(s)v_i - P_i(s) \end{aligned} \tag{17}$$

$$\begin{aligned} &h_i(\phi_i(s))X_i(s)s - P_i(s) \\ \geq &h_i(\phi_i(v_i))X_i(v_i)s - P_i(v_i) \end{aligned} \tag{18}$$

Combining inequality (17) and (18) gives

$$\begin{aligned} &\left[h_i(\phi_i(v_i))X_i(v_i) - h_i(\phi_i(s))X_i(s)\right] v_i \\ \geq &P_i(v_i) - P_i(s) \\ \geq &\left[h_i(\phi_i(v_i))X_i(v_i) - h_i(\phi_i(s))X_i(s)\right] s. \end{aligned} \tag{19}$$

It follows that

$$[h_i(\phi_i(v_i))X_i(v_i) - h_i(\phi_i(s))X_i(s)](v_i - s) \geq 0, \tag{20}$$

which implies $h_i(\phi_i(v_i))X_i(v_i)$ is monotone non-decreasing with respect to v_i.

To satisfy the IR constraint, a feasible mechanism should guarantee that a zero-valued advertiser also has a non-negative utility:

$$U_i(\mathcal{M}, 0) = 0 \cdot h_i(\phi_i(0)X_i(0) - P_i(0) = -P_i(0) \geq 0.$$

However, as we require $P_i(v_i) \geq 0$, It follows that $P_i(0) = 0$ and $U_i(\mathcal{M}, 0) = 0$.

The IC property means that advertiser i's utility is maximized by report v_i, i.e.,

$$U_i(\mathcal{M}, v_i) = \max_{t \in \mathcal{V}} \left\{ v_i h_i(\phi_i(t))X_i(t) - P_i(t) \right\}.$$

According to the envelope theorem [Carter, 2001], we get

$$\begin{aligned} U_i(\mathcal{M}, v_i) &= U_i(\mathcal{M}, 0) + \int_0^{v_i} X_i(s)h_i(\phi_i(s))\,\mathrm{d}s \\ &= \int_0^{v_i} X_i(s)h_i(\phi_i(s))\,\mathrm{d}s \end{aligned}$$

Since $U_i(\mathcal{M}, v_i) = h_i(\phi_i(v_i))v_iX_i(v_i) - P_i(v_i)$, we have

$$P_i(v_i) = h_i(\phi_i(v_i))v_iX_i(v_i) - \int_0^{v_i} h_i(\phi_i(s))X_i(s)\,\mathrm{d}s.$$

Now we show that satisfying the three conditions leads to a feasible mechanism. Similarly, we still only prove the first two conditions, as the third condition is obvious.

For any advertiser i with valuation v_i, the utility gain from reporting a different v_i' is

$$\begin{aligned} &v_ih_i(\phi_i(v_i'))X_i(v_i') - P_i(v_i') - U_i(\mathcal{M}, v_i) \\ =&P_i(v_i) - P_i(v_i') \\ &+ v_i\left[h_i(\phi_i(v_i'))X_i(v_i') - h_i(\phi_i(v_i))X_i(v_i))\right] \\ =&\int_{v_i}^{v_i'} h_i(\phi_i(s))X_i(s)\,\mathrm{d}s + (v_i - v_i')h_i(\phi_i(v_i'))X_i(v_i') \\ =&\int_{v_i}^{v_i'} [h_i(\phi_i(s))X_i(s) - h_i(\phi_i(v_i'))X_i(v_i')]\,\mathrm{d}s \end{aligned}$$

Since $h_i(\phi_i(v_i))X_i(v_i)$ is monotone non-decreasing, it is easy to check that no matter $v_i' > v_i$ or $v_i' < v_i$, the utility gain is non-positive. □

A.2 Proof of Lemma 2

Proof. We can rewrite the platform's objective function as

$$\begin{aligned} \textbf{Rev} =& \sum_{i=1}^n \int_{\mathbf{v}} \left[p_i(\mathbf{b}) - x_i(\mathbf{b})h_i(\phi_i(b_i))r_i(\phi_i(b_i))\phi_i(b_i)\right]\,\mathrm{d}\mathbf{v} \\ =& \sum_{i=1}^n \int_{\mathbf{v}} h_i(\phi_i(v_i))x_i(\mathbf{v})v_if(\mathbf{v})\,\mathrm{d}\mathbf{v} \\ &+ \sum_{i=1}^n \int_{\mathbf{v}} (p_i(\mathbf{v}) - h_i(\phi_i(v_i))x_i(\mathbf{v})v_i)f(\mathbf{v})\,\mathrm{d}\mathbf{v} \\ &- \sum_{i=1}^n \int_{\mathbf{v}} x_i(\mathbf{v})h_i(\phi_i(v_i))r_i(\phi_i(v_i))\phi_i(v_i)f(\mathbf{v})\,\mathrm{d}\mathbf{v} \end{aligned} \tag{21}$$

Using Lemma 1, we know that for any feasible mechanism $\mathcal{M}$:

$$\begin{aligned} &\int_{\mathbf{v}} (p_i(\mathbf{v}) - h_i(\phi_i(v_i))x_i(\mathbf{v})v_i)f(\mathbf{v})\,\mathrm{d}\mathbf{v} \\ =& -\int_0^{\overline{v}} U_i(\mathcal{M}, s)f_i(s)\,\mathrm{d}s \\ =& -\int_0^{\overline{v}} \int_0^s h_i(\phi_i(t))X_i(t)\,\mathrm{d}tf_i(s)\,\mathrm{d}s \\ =& -\int_0^{\overline{v}} (1 - F_i(t))h_i(\phi_i(t))X_i(t)\,\mathrm{d}t \\ =& -\int_{\mathbf{v}} \left[\frac{1 - F_i(v_i)}{f_i(v_i)}\right] h_i(\phi_i(v_i))x_i(\mathbf{v})f(\mathbf{v})\,\mathrm{d}\mathbf{v} \end{aligned} \tag{22}$$

Inserting (22) into (21) gives:

$$\begin{aligned} \textbf{Rev} &= \sum_{i=1}^n \int_{\mathbf{v}} J_i(v_i)x_i(\mathbf{v})f(\mathbf{v})\,\mathrm{d}\mathbf{v} \\ &= \int_{\mathbf{v}} \left[\sum_{i=1}^n J_i(v_i)x_i(\mathbf{v})\right] f(\mathbf{v})\,\mathrm{d}\mathbf{v} \end{aligned}$$

□

B Ommitted Proofs in Section 3

B.1 Proof of Lemma 3

Proof. By definition, function $\varphi_i(c_i, v_i)$ satisfies

$$\varphi_i(c_i', v_i') - \varphi_i(c_i, v_i') \geq \varphi_i(c_i', v_i) - \varphi_i(c_i, v_i). \tag{23}$$

Plugging $\varphi_i(c_i, v_i) = [\kappa_i(v_i) - r_i(c_i)c_i]h_i(c_i)$ into the above equation, we obtain:

$$[\kappa_i(v_i') - \kappa_i(v_i)][h_i(c_i') - h_i(c_i)] \geq 0.$$

The assumption that function $h_i(\cdot)$ is strictly increasing implies that $h_i(c_i') - h_i(c_i) > 0$. In order for Equation (23) to hold, we must have $\kappa_i(v_i') - \kappa_i(v_i) \geq 0$. This further implies that for all c_i,

$$\varphi_i(c_i, v_i') - \varphi_i(c_i, v_i) = [\kappa_i(v_i') - \kappa_i(v_i)]h_i(c_i) \geq 0. \quad (24)$$

According to Topkis's theorem [Topkis, 1978], we have $\phi_i(v_i') \geq \phi_i(v_i)$ for any $v_i' \geq v_i$. Therefore, we have

$$\varphi_i(\phi_i(v_i'), v_i') \geq \varphi_i(\phi_i(v_i), v_i') \geq \varphi_i(\phi_i(v_i), v_i), \quad (25)$$

where the first inequality is due to the definition of $\phi_i(v_i')$ ($\phi_i(v_i')$ is a maximizer of function $\varphi_i(\cdot, v_i')$), and the second inequality is because of Equation (24). Since $J_i(v_i) = \varphi_i(\phi_i(v_i), v_i)$, Equation (25) directly implies that $J_i(v_i)$ is a monotone non-decreasing function. □

B.2 Proof of Lemma 4

Proof. In the proof of Lemma 3, we already show that if function $\varphi_i(c_i, v_i)$ has increasing differences, then $\kappa_i(v_i)$ is a monotone non-decreasing function (see Equation (24)).

Now we show that if $\kappa_i(v_i)$ is monotone non-decreasing, function $\varphi_i(c_i, v_i)$ must have increasing differences. Note that for any $c_i' > c_i$, $h_i(c_i') - h_i(c_i) > 0$. This means for any $v_i' > v_i$, we have

$$\kappa_i(v_i')[h_i(c_i') - h_i(c_i)] \geq \kappa_i(v_i)[h_i(c_i') - h_i(c_i)].$$

Adding the term $r_i(c_i)h_i(c_i)c_i - r_i(c_i')h_i(c_i')c_i'$ to both sides of the inequality immediately yields:

$$\varphi_i(c_i', v_i') - \varphi_i(c_i, v_i') \geq \varphi_i(c_i', v_i) - \varphi_i(c_i, v_i).$$

□

B.3 Proof of Lemma 5

Proof. Since mechanism (ϕ, x, p) is feasible, according to Lemma 2, the revenue of this mechanism can be written as:

$$\begin{aligned}
&\int_{\mathbf{v}} \left[\sum_{i=1}^{n} J_i(v_i)x_i(\mathbf{v})\right] f(\mathbf{v})\,\mathrm{d}\mathbf{v} \\
=&\int_{\mathbf{v}} \left[\sum_{i=1}^{n} \overline{\varphi}_i(\phi_i(v_i), v_i)x_i(\mathbf{v})\right] f(\mathbf{v})\,\mathrm{d}\mathbf{v} \\
&+ \int_{\mathbf{v}} \left[\sum_{i=1}^{n} \Big(\varphi_i(\phi_i(v_i), v_i) - \overline{\varphi}_i(\phi_i(v_i), v_i)\Big)x_i(\mathbf{v})\right] f(\mathbf{v})\,\mathrm{d}\mathbf{v}
\end{aligned} \quad (26)$$

Then consider the second term of the right-hand side.

$$\begin{aligned}
&\int_{\mathbf{v}} \left[\sum_{i=1}^{n} [\varphi_i(\phi_i(v_i), v_i) - \overline{\varphi}_i(\phi_i(v_i), v_i)]\, x_i(\mathbf{v})\right] f(\mathbf{v})\,\mathrm{d}\mathbf{v} \\
=&\int_{\mathbf{v}} \left[\sum_{i=1}^{n} [\kappa_i(v_i) - \overline{\kappa}_i(v_i))\, h_i(\phi_i(v_i))]x_i(\mathbf{v})\right] f(\mathbf{v})\,\mathrm{d}\mathbf{v} \\
=&\int_{\mathbf{v}} \left[\sum_{i=1}^{n} [k_i(F_i(v_i)) - \ell_i(F_i(v_i))]h_i(\phi_i(v_i))x_i(\mathbf{v})\right] f(\mathbf{v})\,\mathrm{d}\mathbf{v} \\
=&\sum_{i=1}^{n} \left[\int_{\mathbf{v}} [k_i(F_i(v_i)) - \ell_i(F_i(v_i))]\, h_i(\phi_i(v_i))x_i(\mathbf{v})\right] f(\mathbf{v})\,\mathrm{d}\mathbf{v}
\end{aligned} \quad (27)$$

Combining Equation (27) and (26) immediately leads to Equation (13). □

B.4 Proof of Theorem 2

Proof. We first show the feasibility of the mechanism. During the ironing procedure, it is clear that the function $L_i(q_i)$ is convex. As the derivative of $L_i(q_i)$, $\ell_i(q_i)$ is then monotone non-decreasing. Both $F_i(v_i)$ and its inverse function $F_i^{-1}(q_i)$ are increasing functions. Therefore, after the variable substitution, $\overline{\kappa}_i(v_i)$ is also a non-decreasing function. This guarantees that the mechanism constructed based on $\overline{\kappa}_i(v_i)$ is feasible. The proof is the same as that of Theorem 1, and thus omitted here.

We claim that the mechanism $\overline{\mathcal{M}}$ is optimal, i.e., optimizes Equation (13). After applying the ironing procedure, the function $\overline{\varphi}_i(\phi_i(v_i), v_i)$ has increasing differences. The first term of Equation (13) can be optimized using the same method as in the regular case. With the same argument as in the proof of Theorem 1, we know that mechanism $\overline{\mathcal{M}}$ maximizes the first term of Equation (13).

We now show that mechanism $\overline{\mathcal{M}}$ also optimizes the second term of Equation (13).Using integration by parts, we derive the following:

$$\begin{aligned}
&\int_{\mathbf{v}} [k_i(F_i(v_i)) - \ell_i(F_i(v_i))]\, x_i(\mathbf{v})f(\mathbf{v})\,\mathrm{d}\mathbf{v} \\
=&\int_0^{\overline{v}} [k_i(F_i(v_i)) - \ell_i(F_i(v_i))]\, h_i(\phi_i(v_i))X_i(v_i)f_i(v_i)\,\mathrm{d}v_i \\
=&[K_i(F_i(v_i)) - L_i(F_i(v_i))h_i(\phi_i(v_i))X_i(v_i)]\Big|_0^{\overline{v}} \\
&- \int_0^{\overline{v}} [K_i(F_i(v_i)) - L_i(F_i(v_i))]\,\mathrm{d}\Big[h_i(\phi_i(v_i))X_i(v_i)\Big] \\
=&\int_0^{\overline{v}} [L_i(F_i(v_i)) - K_i(F_i(v_i))]\,\mathrm{d}\Big[h_i(\phi_i(v_i))X_i(v_i)\Big] \\
\leq& 0,
\end{aligned}$$

where the last equality follows since L_i is the convex hull of the graph of K_i on $[0, 1]$ and K_i is continuous, implying that $K_i(0) = L_i(0)$ and $K_i(1) = L_i(1)$. The last inequality follows from the facts that $L_i \leq K_i$ by construction and for any feasible direct mechanism $h_i(\phi_i(v_i))X_i(v_i)$ is monotone non-decreasing.

Note that during the ironing procedure for any $q_i \in [0, 1]$, there are only two cases

- $L_i(q_i) = K_i(q_i)$. In this case, the integrand $L_i(q_i) - K_i(q_i)$ is 0.
- $L_i(q_i) < K_i(q_i)$. In this case, there must exist a starting point $q_i^1 < q_i$ and $q_i^2 > q_i$, such that $K_i(q_i^1) = L_i(q_i^1)$, $K_i(q_i^2) = L_i(q_i^2)$ and $K_i(q_i') < L_i(q_i'), \forall q_i' \in (q_i^1, q_i^2)$. It follows from the ironing procedure that $\overline{\kappa}_i(v_i) = \ell_i(q_i) = \frac{K_i(q_i^2) - K_i(q_i^1)}{q_i^2 - q_i^1}$ is a constant in the interval $[q_i^1, q_i^2]$. As a result, $h_i(\phi_i(v_i))X_i(v_i)$ is also a constant in the interval.

Since it is always the case that either $L_i(q_i) - K_i(q_i) = 0$ or $\mathrm{d}\Big[h_i(\overline{\phi}_i(v_i))\overline{X}_i(v_i)\Big] = 0$, we have:

$$\int_0^{\overline{v}} [L_i(F_i(v_i)) - K_i(F_i(v_i))]\,\mathrm{d}\Big[h_i(\overline{\phi}_i(v_i))\overline{X}_i(v_i)\Big] = 0.$$

Therefore, mechanism $\overline{\mathcal{M}}$ also maximizes the second term of Equation (13), and thus simultaneously maximizes both terms, which means mechanism $\overline{\mathcal{M}}$ is optimal.

□

C Omitted Proofs in Section 4

C.1 Proof of Lemma 6

Proof. Recall that in the optimal mechanism, the coupon function $\bar{\phi}_i(v_i)$ maximizes $\bar{\varphi}_i(c_i, v_i) = h_i(c_i)[\overline{\kappa}_i(v_i) - r_i(c_i)c_i] = h_i(c_i)\overline{\kappa}_i(v_i) - h_i(c_i)r_i(c_i)c_i$ for any given v_i. Clearly, the second term $h_i(c_i)r_i(c_i)c_i$ is a non-decreasing function. If $\overline{\kappa}_i(v_i) \leq 0$, the first term become non-increasing, making $\bar{\varphi}_i(c_i, v_i)$ a non-increasing function of c_i. This means setting $c_i = 0$ maximizes $\bar{\varphi}_i(c_i, v_i)$. Thus, we have $\bar{\phi}_i(v_i) = 0$, i.e., the optimal mechanism offers no coupon in this case. □

C.2 Proof of Lemma 7

Proof. The coupon function $\bar{\phi}_i(v_i)$ should maximize $\bar{\varphi}_i(c_i, v_i)$ in the optimal mechanism. If $\bar{\phi}_i(v_i) > 0$, according to first-order condition, we have:

$$
\begin{aligned}
&\left.\frac{\partial \bar{\varphi}_i(c_i, v_i)}{\partial c_i}\right|_{c_i=\bar{\phi}_i(v_i)} \\
&= \left.\left[\bar{\kappa}_i(v_i)\frac{\mathrm{d}}{\mathrm{d}c_i}h_i(c_i) - \frac{\mathrm{d}}{\mathrm{d}c_i}[h_i(c_i)r_i(c_i)c_i]\right]\right|_{c_i=\bar{\phi}_i(v_i)} = 0.
\end{aligned}
$$

which completes the proof. □